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Hume's Hist. vol. vii. p. 55. 2^o edit. 1776.

Engraved from a drawing by McBrown in the possession of the Earl of Buchan



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No. 6.

AN
ACCOUNT
OF THE
LIFE, WRITINGS, and INVENTIONS
OF
JOHN NAPIER,
OF
MERICSTON;
BY
DAVID STEWART, EARL OF BUCHAN,
AND
WALTER MINTO, L. L. D.
ILLUSTRATED WITH COPPERPLATES.

QUANDO ULLUM INVENIES PAREM?

PERTH:
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M,DCC,LXXXVII.

1890. JULY 1.

TO THE
K I N G.

SIR,

AS the writings of Archimedes were addressed to the King of Sicily, who had perused and relished them, so I do myself the honour, to address to Your Majesty, the following account of the Life, Writings, and Inventions of our British Archimedes, in which, I can claim no other merit, than having endeavoured to call forth and illustrate the abilities of others. I feel great pleasure, in dedicating this Tract to Your Majesty, after the chaste and dignified model of Antiquity, bestowing on the King, the merited encomium, of having promoted the Sciences and Arts, with which it is connected; and in assuring Your Majesty, that I am, with the greatest respect,

Your MAJESTY'S

Most dutiful Subject, and

Obedient humble Servant,

BUCHAN.



BRITISH

E R R A T A.

Pag. lin.

20 1 for Busbane, read Brisbane,
26 7 for proposition, read proportion,
27 3 for *penitas* read *penitus*
— 6 for *pristinia* read *pristina*
31 20 for 6 and 6' read *b* and *b'*
66 21 for members, read numbers,
71 22 for fewer read power
75 8 for *u* read μ
77 17 for *u* read μ — lin. 22 ditto
113 in the Table of Log. [N° 26.] fig. 23. for 1 read 7 and
for 7 read 1 — It is intreated, that as the Table has no
other errors, these may be corrected with the pen of every
reader or purchaser of this book.

А Т А Я Я П

Лирическое произведение
в прозе
автора Гоголя Николая Васильевича
написано в 1842 году
и впервые опубликовано в журнале
«Современник» в 1843 году.

Составлено из писем, посланных
автором в 1842 году в Петербург
и в Москву, а также из записок
о поездках в Крым и в Сибирь.

LIST OF BOOKS,

QUOTED or CONSULTED, to ELUCIDATE

THE

LIFE AND WRITINGS,

OF

JOHN NAPIER

OF

MERCHISTON.

ARCHIMEDIS Syracusani Arenarius, &c. Eutocii Ascalonitæ in hanc comment. cum versione et notis Joh. Wallis. Dictionnaire Historique et Critique par M. Pierre Bayle. A Rotterdam, 1720. Folio paffim. Balcarres' Memoirs. Bernoulli Ars conjectandi et tractatus de seriebus infinitis. Bafileæ, 1713. 4to. — Opera Omnia. Lausannæ et Genevæ. 1742.

Biographia Britannica.

Boscovich de Cycloide et Logistica.

Arithmetica Logarithmica five logarithmorum chiliades triginta; pro numeris naturali serie crescentibus ab unitate ad 20.000 et a 90.000 ad 100.000: quorum ope multa perficiuntur arithmeticæ problemata et geometrica. Hos numeros primus invenit clarissimus vir Johannes Neperus Baro Merchistonii: eos autem ex ejusdem sententia mutavat, eorumque ortum et usum illustravit Henricus Briggius, in celeberrima Academia Oxoniensi Geometriæ Pro-

fessor Savilianus. Deus nobis usuram vitæ dedit et ingenii, tanquam pecunia nula præstituta die. Londini 1624. Folio.

Boethius de Arithmetica.

Causaboni Epistolæ.

Christophori Clavii Bambergensis, e Soc. Jesu, Opera Mathematica. Moguntiæ 1611. Folio.

— de Astrolabio. Vol. III.

Chambers' Dictionary, 2 vols Folio.

Craufurd's Peerage of Scotland.

— lives of the officers of State.

Crugerus Pref. in Præxin Trig.

Nouveau Dict. Hist. et Crit. pour servir de supplément au Dict. de M. Bayle, par Jaques George de Chauffepie. Amsterdam, 1756. Folio paffim.

Douglas's Peerage of Scotland.

Duteus inquiry.

Exercitationes Geometricæ. Aut. Jacob. Gregory. 1668.

Herwarti ab Hohenburgh opera. 1610.

Mathematical Tables containing Common, Hyperbolic and Logistic logarithms; also sines, Tangents

tangents, secants and versed sines, both natural and logarithmic, &c. to which is prefixed a large and original history of the discoveries and writings relating to those subjects &c. By Charles Hutton, LLD. F. R. S. and Prof. Math Royal Acad. Woolwich. Lond. 1785.

Humes History of the Stuarts.

Keill de Log. TACONIUS or 1714
Joannes Keppleri aliorumque Epistolæ mutuæ.
Lipsiæ 1718. Folio passim.

— Ephemerides novæ motuum Coelestium ab anno 1617.

Leyburnes Recreations. Folio. 1694.

Lilly's Life. London, 1721. 8vo.

Moreland Sir Samuel.

Histoire des Mathematiques par M. Montucla de l'Academie Royale des Sciences & Belles-Lettres de Prusse. 2 tomes quarto. a Paris 1758. passim.

Newtoni Principia. Amst. 1723.

Phil. Transact. London.

Nicolai Raymari Ursi Dithmarsi Fundamentum Astronomicum, id est, nova doctrina sinnuum et triangulorum eaque absolutissima et perfectissima ejusque usus in astronomica calculatione et observatione. Argentorati. 1588. 4to.

Vitæ quorundam erudissimorum et illustrium virorum, scriptore Thoma Smith. Londini 1707. Commentariolus de vita et scriptis D. Henrici Briggii.

Stifellii Arithmetica integra.

Tabulæ Rudolphinæ, quibus Astronomicæ scientiæ temporum longinquitate collapsæ restauratio continetur; a Phœnico illo astronomicum Tychone, ex illustri et generosa Braheorum in regno Daniae familia oriundo equite, &c. &c. curante Joanni Keplero. Ulmæ, anno 1627. Folio.

Sir Thomas Urquhart of Cromertie's Trifolietras. London, 1650. 4to.

— Tracts. Edinburgh, 1774. 8vo.

Vofilius de Mathemat. Reid's Essay on the Log.

Wallisii Opera.

Oughheds' Clavis Mathematica. Oxford, 1677. 8vo.

Worcester Marquis of, his Scantlings of modern Inventions.

Craig's quadrature of the logarithmic curve.

Rabdologiæ seu numerationis per virgulas libri duo. Edinburgi, 1617. 12mo.

A plain discovery of the whole Revelation of St John, &c. by John Napier of Merchiston. Edinburgh, 1593. 4to. by Andro Hart.

Mirifici Logarithmorum Canonis descriptio &c, Authore Joanne Nepero Barone Merchistonii Edinburgæ 1614 apud And. Hart. Imitatio Nepeirea a Henrico Briggs. MSS. 1614. in the British Museum.

Ayscough's Cat. of the MSS in the British Museum. London. 2 vols 4to.

Perault des hommes illustres.

Tychonis Brahæ vita, Gassendo Authore. Parisiis 1654. 4to.

Pitisci Trigonometria.

The lives of the Professors of Gresham College by John Ward, Prof. of Rhetoric in Gresham College. F. R. S. London 1740. Folio.

Logarithmotechnia sive methodus construendi Logarithmos nova accurata et facilis; scripto antehac communicato, anno sc. 1667 nonis Augusti: cui nunc accedit vera quadratura hyperbolæ et inventio summae logarithmorum, Auctore Nicolas Mercatore Holsato, e Societate Régia Londini. 1668.

Developpement Nouveau de la partie élémentaire des Mathematiques prise dans toute son étendue: par Louis Bertrand. 2 tomes Geneve, 1778. 4to.

Trigonometria Britannica. Goudæ 1633. Fol. Memoires de Mathematique et de Physique de l'année 1756. tirees des registres de l'Academie Royale des Sciences a Amsterdam 1768. La trigonometric sphérique reduite a quatre analogies par M. Pingre.

Sherwin's Tables. 1771.

Encyclopédie ou Dictionnaire raisonné des sciences des arts et des métiers. Neufchâtel, 1765. Fol. passim.

Rogerii Cotesii Harmonia Mensurarum. Cantabrigiæ, 1722.

Abridgement of the Philosophical Transactions.

ons by Lowthorp et Motte. 5 vols 4to.
paſſim.

Athenæ Oxonienses, by Anthony Wood. 2
vols Folio. London, 1691.

Woods Hist. et Ant. Oxon. 2 vols Folio.
Oxonie, 1724.

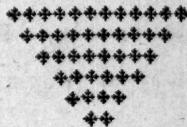
Tables portatives de Logarithmes, publiées
a Londres par Gardiner, augmentées et per-
fectionnées dans leur disposition par M. Cal-
let, et corrigées avec la plus scrupuleuse
exactitude : contenant les logarithmes des
nombres depuis 1 jusqu'à 102960, les lo-
logarithmes des sinus et tangentes, de
ſeconde en ſeconde pour les deux premiers
degrés et de 10 en 10 ſecondes pour tous
les degrés du quart de cercle ; precedées
d'un précis élémentaire ſur l'explication et
l'usage des logarithmes et ſur leur applica-
tion aux calculs d'intérêts, à la Géome-

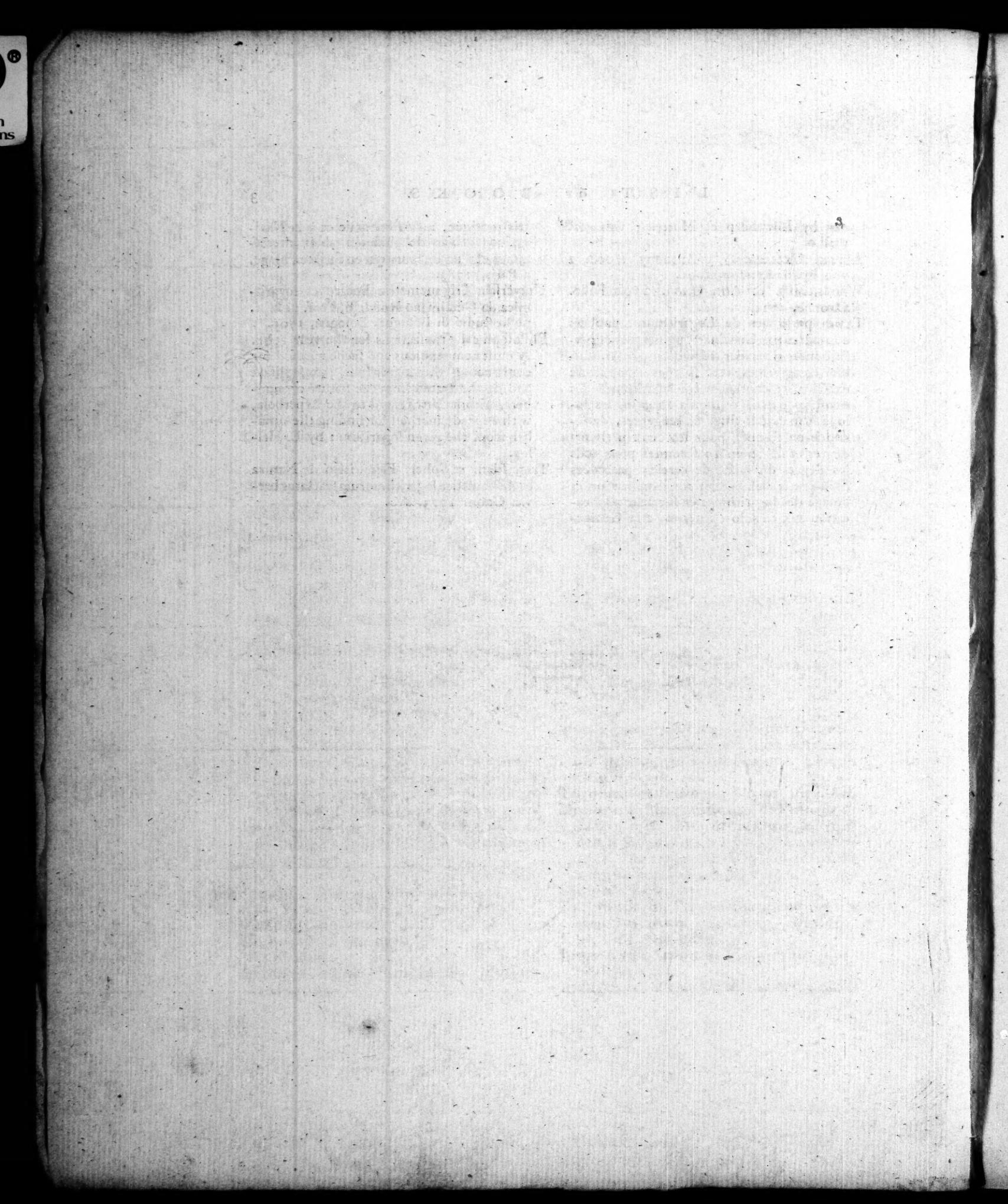
trie-pratique, à l'Aſtronomie et à la Navi-
gation ; ſuivies de pluſieurs tables interef-
fantes et d'un diſcours qui en facilite l'ufage.
a Paris, 1783.

Universale Trigonometria lineare et logarit-
mica da Geminiano Rondelli, Prof. di Mat.
nello studio di Bologna. Bologna, 1705.

Philosophical Transactions for the year 1695.
A most compendious and facile method for
constructing the logarithms, exemplified
and demonstrated from the nature of num-
bers, without any regard to the hyperbola,
with a ſpeedy method for finding the num-
ber from the given logarithm : By E. Hal-
ley.

Trig. Plan. et Spher. Elem. item de Natura
et Arithmeticalogarithmorum tractatus bre-
vis. Oxon. 1723. 8vo.





ADVERTISEMENT.

ABOUT twenty years ago, I thought it would be easy to bring together a group of learned men, who would dedicate a part of their leisure to erect literary monuments to the memory of their illustrious countrymen, whose lives had not been hitherto written or sufficiently illustrated; and I wished such monuments to be fashioned and executed by men personally eminent in the departments which distinguished the subjects of their biographical research, and not by the assistants of a bookseller or compiler, who cannot be expected, however faithful and accurate, to be animated with that love to the subject, which the Italian Artist rightly considers as the soul of his enterprize, and the source of its perfection.

In this expectation I have been disappointed; and though I allow the highest merit to the British Biography, now republishing by Dr Kippis, yet in the immense extent of such an undertaking, I perceive the impossibility of its reaching the perfection I have proposed, without the addition of supplementary articles and connections, which would have been in a great measure unnecessary, had my plan been adopted; because the articles, being written with care and with zeal, so as to support themselves in an isolated state by the public favour, would afterwards have been taken up by subsequent editions into that great repository of biographical learning, in a highly finished state, and purged of the errors which are unavoidable, in the first fabric of works of that nature.

With respect to the biography of Scotland, one of the judges there, who would have done it honour in its best days, by his virtue, his attention to the dignity and duties of his station, and the useful employment of his leisure, has generously offered, by an advertisement annexed to the *Annals of the Lives of John Barclay, Author of Argenis, and some other learned Scots*, to forward the undertaking I wish to promote.

Encouraged by the assistance of an associate, so able and so liberal, I have presumed to offer the following Biographical Tract to the public, as my mite to a Treasury, which I hope to see enriched by many, who have the ability and the generosity of my respectable coadjutor. It was indeed by that excellent man, that I was originally encouraged to prosecute researches of this nature. He applauded that disposition in a young man of quality, which leads him to the study of the history of his own country, not in pamphlets, satires, apologies and panegyrics, but in the private undisguised correspondence of the great.

A man who studies history in this way, will see that the same characters are represented by different actors: introduced behind the scenes, he will see folly dressing itself in the garb of wisdom, and selfishness assuming the mask of public spirit; and among the learned, the plagiary stealing away the laurels of the modest inventor. He will see great events arising from inconsiderable causes, and men neither devils nor angels, but a composition of good and bad qualities, such as the men of the world can see them every day in common life.

I flatter myself, that this article of Napier, in the *Biographia Scotica*, will be considered in some respects, as a specimen of the plan I have described, for it certainly has been written con amore. In the scientific part I have received the assistance of a gentleman, who deserves to be better known, on account of his mathematical learning, and the accuracy with which he treats the subjects of his inquiry.

If the following publication, shall have the good fortune to meet with the approbation of the learned world, 'tis my intention, to give an account of the lives and writings of Andrew Fletcher of Salton, and John Law of Laurieston, on the same plan. The first undertaking will furnish me an opportunity, of representing the ancient constitution of Scotland, in what I apprehend to be a clearer light, than has hitherto been offered; and of treating the causes and consequences of the union between the two kingdoms: and the other will open an ample field for exhibiting the disorders in the finances of France, occasioned by the expensive wars of Lewis the fourteenth, and the Mississippi Scheme, and for explaining by what means they have been gradually remedied and brought to a state, which has enabled that nation, not only to bring her naval force and her trade to a dangerous rivalry with this country, but to obtain that credit, by good faith, which in former times, had given so decided a superiority to Britain. I am very sensible that there are many men in this country much better qualified for performing these tasks than I am, and I think it an honour to enjoy their friendship: but men of great reputation generally seek for rest in the evening of life, and avoid exposing their laurels to the blast of envy, in their declining years.

These, I hope, will be accepted as sufficient apologies, for my venturing to occupy such ground, and I beg leave to invite my learned countrymen, to aid me in so noble an undertaking, as that of raising monuments to the memory of the illustrious dead.

I have only to add, that if the separate lives of illustrious persons, should be written on the plan I propose, and were accompanied by portraits, elegantly engraven by the best artists, and the whole executed in a similar manner, of the same Quarto size, and with the same Type and Paper, they would gradually form the noblest work, which has been offered to the republic of letters, in any age or country.

an English reader has ~~arrived~~ ^{arrived} to be malignant but with soft
and benevolent feelings, and to be a friend to the principles of
Truth.

ACCOUNT

of the life of JOHN NAPIER, of Merchiston, a man famous
all the world over, for his great and fortunate discov-

ery of Logarithms in Trigonometry, by which the ease and expedition
in calculation, have so wonderfully assisted the Science of Astronomy,
LIFE, WRITINGS, and INVENTIONS,

JOHN NAPIER,

MERCHISTON.

I HAVE undertaken to write the Life of JOHN NAPIER, of Merchiston, a man famous all the world over, for his great and fortunate discovery of Logarithms in Trigonometry, by which the ease and expedition in calculation, have so wonderfully assisted the Science of Astronomy, and the arts of practical Geometry and Navigation.

ELEVATED above the age in which he lived, and a benefactor to the world in general, he deserves the epithet of *Great*.

NAPIER lived in a country of proud Barons, where barbarous hospitality, hunting, the military art, and religious controversy, occupied

the time and attention of his contemporaries, and where he had no learned society to assist him in his researches.

T H U O D D A

THIS extraordinary person was born at Merchiston, in the neighbourhood of Edinburgh, in the year 1550*.

He was the Son of Sir Archibald Napier, of Merchiston, Master of the Mint in Scotland, and of Janet Bothwell, daughter of Mr Francis Bothwell, one of the Senators of the college of Justice †.

THAT his family was of ancient establishment in the counties of Dunbarton and Stirling, appears from the public records, and from the private archieves of his house.

JOHN de Napier, from whom he sprung in the 12th generation, was one of those proprietors of lands, who swore allegiance to Edward the first, of England, in the year 1296. William, from whom he counted in the ninth generation, was Governor of the Castle of Edinburgh, in the year 1401, whose son Alexander, was the first Baron or Laird of Merchiston, and was the Father of another of the same name, who was Vice Admiral of Scotland, and one of the Commissioners from king James the third, at the court of London, in the years 1461 and 1464.

FROM the family of Lennox, Earl of Lennox, he derived a coheirship by the marriage of Elizabeth Mentieth, of Rusky, to his great-grand-

* As appears by an inscription on his portrait, engraved by old Cooper, from an original painting.

† Craufurd's Peerage.

father's father, Sir John Napier, of Merchiston: but on his ancestors he reflected more honour and celebrity than he received, and his name will probably be famous, when the lineage of Plantagenet will be remembered only by genealogists, and when posterity may know no more of his, than we now know of the families of Plato, Aristotle, Archimedes, or Euclid.

It is fit, that men should be taught to aim at higher and more permanent glory than wealth, office, titles or parade can afford; and I like the task, of making such great men look little, by comparing them with men who resemble the subject of my present enquiry.

FROM Napier's own authority, we learn, that he was educated at St. Andrews *, where writes he, "in my tender years and bairn-age, at schools, having on the one part contracted a loving familiaritie with a certain gentleman a papist, and on the other part being attentive to the sermons of that worthy man of God, Maister Christopher Goodman, teaching upon the Apocalyps, I was moved in admiration against the blindnes of papists that could not most evidentlie see their seven hilled Citie of Rome, painted out there so lively by Saint John, as the Mother of all Spiritual Whoredome: that not onlie bursted I oute in continuall reasoning against my said familiar, but also from thenceforth I determined with myself by the affistance of God's spirit to employ my study and diligence to search out the remanent mysteries of that holy booke (as to this hour praised be the Lord I have bin doing at all such times as convenientlie I might have occasion) &c.

THE

* Preface to his plain discovery of the Revelation of St. John.

THE time of Napier's matriculation does not appear from the Register of the University of St. Andrews, as the books ascend no higher than the beginning of the last century; but as the old whore of Babylon, assumed in the eyes of the people of Scotland, her deepest tinge of scarlet about the year 1566, and as that time corresponds to the literary bairn-age of John Napier, I suppose, he then imbibed the holy fears and commentaries of Maister Christopher Goodman, and as other great Mathematicians have ended, so he began his career with that mysterious book.

I have not been able to trace Merchiston from the University, till the publication of his Plain Discovery, at Edinburgh, in the year 1593 *; though Mackenzie in his lives and characters of the most eminent writers of the Scotch nation, informs us (without quotation, however, of any authority) that he passed some years abroad, in the low countries, France and Italy, and that he applied himself there, to the study of Mathematics.

IN the British Museum there are two copies of his letter to Anthony Bacon, the original of which, is in the Archbishop's Library at Lambeth, entitled "Secret Inventions, profitable and necessary, in these days, for the defence of this Island, and notwithstanding strangers enemies to God's truth and religion," which I have caused to be printed, in the Appendix to this Tract. This letter is dated, June 7, 1596 †, about which time it appears, as shall be shewn hereafter, that he had set himself to explore his Logarithmic Canon.

I

* Printed by Andrew Hart, 4to. † Ayscough lat. v. i. p. 155. See Appendix N^o. 1.

I have enquired, without success, among all the descendants of this eminent person, for papers or letters, which might elucidate this dark part of his history; and if we consider that Napier was a recluse mathematician, living in a country, very inaccessible to literary correspondence, we have not much room to expect, that the most diligent explorations, would furnish much to the purpose, of having the progress of his studies.

AMONG Mr Briggs's papers, preserved in the British Museum, I looked for letters from Napier, but found only what Mr Briggs calls, his *Imitatio Nepeirea, sive applicatio omnium fere regularum, suis Logarithmis pertinentium, ad Logarithmos*; which seems to have been written in the year 1614, soon after the publication of the *Canon**.

THOUGH the life of a learned man is commonly barren of events, and best unfolded in the account of his writings, discoveries, improvements, and correspondence with the learned men of his age, yet I anxiously sought for somewhat more, with respect to a character, I so much admired; but my researches have hitherto been fruitless. Perhaps from the letters, books, and collections of societies or of learned individuals, to which I have not had access, something may hereafter be brought to light: and one of the inducements, to offer a sketch of this kind to the public, is the tendency it may have to bring forth such information. His plain discovery has been printed abroad, in several languages, particularly in French, at Rochelle, in the year 1603, 8vo. an-

D
nounced

* Ayscough's Cat. vol. i. p. 389.

nounced in the title, as revised by himself *. Nothing could be more agreeable to the Rochellers, or to the Hugonots of France, at this time, than the Author's annunciation of the Pope, as Antichrist, which in this book he has endeavoured to set forth, with much zeal and erudition.

THAT Napier had begun, about the year 1593, that train of enquiry, which led him to his great atchievement in Arithmetic, appears from a letter to Crugerus from Kepler, in the year 1624; wherein, mentioning the Canon Mirificus, he writes thus, *Nihil autem supra Neperianam rationem esse puto: etsi Scotus quidam literis ad Typhonem, anno 1594, Scriptis jam spem fecit Canonis illius mirifici*, which allusion agrees with the idle story mentioned by Wood in his *Athenæ Oxon*, and explains it in a way perfectly consonant to the rights of Napier as the inventor; concerning which, I shall take occasion to comment, in the account of his works: nor is it to be supposed, that had this noble discovery been properly applied to science, by Justus Byrgius, or Longomontanus, Napier would have been universally acknowledged by his contemporaries, as the undisputed Author of it.

No men in the world, are so jealous of each other as the learned, and the least plausible pretence of this sort, could not have failed to produce

a.

* This edition was published on the first day of that year, in the end of which the Synod of Gap did declare, or moved to declare, the Pope to be Antichrist, which had never been before attempted, by any body of Protestants. See Sully's *Memoirs*.

WITH respect to Napier's fanciful calculation of the completion of the prophecies, concerning the duration of the world, the year, in which this monument is erected to his memory, immediately succeeds that fixed for the end of the world, and no doubt must be the year of judgment, with respect to the authenticity of his discovery, and the merit of those arguments, which are brought forward to support his claim.

a controversy, in the republic of letters, both in his lifetime, and after his death, when his praises were sounded all over Europe *.

WHEN

* To quote authorities in this place, would be to give a catalogue of all the Mathematical and Arithmetical books of that age.

His most outrageous panegyrist, is Sir Thomas Urquhart, of Cromarty, who has given us also so ridiculous an account of the admirable Crichton.

In his Jewel, Urquhart, after having referred his readers to his Trigonometrical Work, entitled Triflottetras, for the praises of Napier, thus mentions " an almost incomprehensible device, which being in the mouths of the most of Scotland, and yet unknown to any that ever was in the world but himself, deserveth very well to be taken notice of in this place ; and it is this : he had the skill, as is commonly reported, to frame an engine, (for invention not much unlike that of Archytas's Dove) which by virtue of some secret springs, inward resorts, with other implements, and materials fit for the purpose, inclosed within the bowels thereof, had the power (if proportionable in bulk to the action required of it (for he could have made it of all sizes) to clear a field of four miles circumference, of all the living creatures exceeding a foot in heighth, that should be found thereon, how near soever they might be found to one another ; by which means he made it appear, that he was able, with the help of this machine alone, to kill thirty thousand Turks, without the hazard of one Christian ! " Of this it is said that (on a wager) he gave proof upon a large plain in Scotland, to the destruction of a great many head of Cattle, and flocks of sheep, whereof some were distant from other half a mile on all-sides, and some a whole mile. To continue the thread of my story, as I have it, I must not forget, that when he was most earnestly desired by an old acquaintance, and professed friend of his, even about the time of his contracting the disease whereof he died, that he would be pleased, for the honour of his family, and his own everlasting memory to posterity, to reveal unto him the manner of the contrivance of so ingenious a mystery, subjoining thereto, for the better persuading him, that it were a thousand pities, that so excellent an Invention should be buried with him in the grave, and that after his decease nothing should be known thereof : his answer was, that for the ruin and overthrow of man, there were too many devices already framed, which if he could make to be fewer, he would with all his might endeavour to do ; and that, therefore, seeing the malice and rancor rooted in the heart of mankind, will not suffer them to diminish the number of them, by any new concert of his should never be increased. Divinely spoken truly.

Urquhart's Tracts, Edinburgh, 1774. 8vo. p. 57.

WHEN Napier had communicated to Mr Henry Briggs, Mathematical Professor in Gresham College, his wonderful Canon for the Logarithms, that learned Professor set himself to apply the rules in his *Imitatio Neptinea*, which I have already mentioned, and in a letter to Archbishop Usher, in the year 1615, he writes thus, “ Napier, Lord of “ Merchiston, hath set my head and hands at work with his new and “ admirable Logarithms. I hope to see him this summer if it please “ God, for I never saw a book which pleased me better, and made me “ more wonder”*.

IT may seem extraordinary to quote Lilly the astrologer with respect to so great a man as Napier; yet as the passage I propose to transcribe from Lilly’s life, gives a picturesque view of the meeting betwixt Briggs and the Inventor of the Logarithms, at Merchiston near Edinburgh, I shall set it down in the original words of that mountebank knave †.

“ I will acquaint you with one memorable story, related unto me by John Marr, an excellent mathematician and geometrician, whom I conceive you remember. He was servant to king James the first and Charles the first. When Merchiston first published his Logarithms, Mr Briggs then reader of the Astronomy Lectures at Gresham College in London, was so surprised with admiration of them, that he could have no quietness in himself, until he had seen that noble person whose only invention they were: He acquaints John Marr therewith, who went into Scotland before Mr Briggs, purposely to be there when these two so learned persons should meet; Mr Briggs appoints a certain day when

to

* Usher’s Letters, p. 36.

† Lilly’s Life, London, 1721. 8vo.

to meet at Edinburgh, but failing thereof, Merchiston was fearful he would not come. It happened one day as John Marr and the Lord Napier were speaking of Mr Briggs; “ Ah John, saith Merchiston, Mr Briggs will not now come”: at the very instant one knocks at the gate; John Marr hasted down and it proved to be Mr Briggs to his great contentment. He brings Mr Briggs up into My Lord’s chamber, where almost one quarter of an hour was spent, each beholding other with admiration before one word was spoken: at last Mr Briggs began. “ My Lord I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help unto Astronomy, *viz.* the Logarithms; but My Lord, being by you found out, I wonder nobody else found it out before, when now being known it appears so easy”. He was nobly entertained by the Lord Napier, and every summer after that during the Laird’s being alive, this venerable man Mr Briggs went purposely to Scotland to visit him.”

THERE is a passage in the life of Tycho Brahe by Gassendi*, which may mislead an inattentive reader to suppose that Napier’s method had been explored by Herwart at Hoenburg, ’tis in Gassendi’s observations on a letter from Tycho to Herwart, of the last day of August 1599. *Dixit Hervartus nihil morari se solvendi cuiusquam trianguli difficultatem; sole*re se enim multiplicationum, ac divisionum vice additiones solum, subtractiones 93 usurpare (quod ut fieri posset, docuit postmodum suo Logarithmorum Canone Neperus.) But Herwart here alludes to his work afterwards published

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* Tychonis Brahei Vita. Parisius 4to, 1654. p. 191.

in the year 1610, which solves triangles by Prosthaphæresis, a mode totally different from that of the Logarithms.

KEPLER dedicated his Ephemerides to Napier, which were published in the year 1617 *; and it appears from many passages in his letter about this time, that he held Napier to be the greatest man of his age, in the particular department to which he applied his abilities: and indeed if we consider, that Napier's discovery was not, like those of Kepler or of Newton, connected with any analogies or coincidences, which might have led him to it, but the fruit of unassisted reason and science, we shall be vindicated in placing him in one of the highest niches in the Temple of Fame.

KEPLER had made many unsuccessful attempts to discover his canon for the periodic motions of the planets and hit upon it at last, as he himself candidly owns, on the 15th of May, 1618; and Newton applied the palpable tendency of heavy bodies to the earth to the system of the universe in general; but Napier sought out his admirable rules, by a slow scientific progress, arising from the gradual revolution of truth.

THE last literary exertion of this eminent person, was the publication of his Rabdology and Promptuary, in the year 1617, which he dedicated to the Chancellor Seton, and soon after died at Merchiston, on the 3d of April, O. S. of the same year, in the 68th year of his age, and, as I suppose, in the 23d of his happy invention.

IN

* Kepler's Ephemerides novæ motuum cælestium ab anno 1617.

IN his person, the portraits * I have seen represent him of a grave and sweet countenance, not unlike his eminent contemporary Monsieur de Peiresc.

IN his family he seems to have been uncommonly fortunate, for his eldest son became learned and eminent even in his father's lifetime, his third a pupil of his own in Mathematics, to whom he left the care of publishing his Posthumous works; and losing none of his children by death, he lost all his daughters by honourable or respectable marriages.

HE was twice married. By his first wife, Margaret, the daughter of Sir James Stirling of Kier, descended of one of the oldest and most respectable gentlemen's families in Scotland, he had an only child, Archibald, his successor in his estates, of whom I shall hereafter give some account. By his second marriage with Agnes, the daughter of Sir James Chisholm, of Crombie, he had five sons: John, Laird of Easter Tonie; Robert †, who published his father's works, whom I have already mentioned, the ancestor of the Napiers of Kilkroigh in Stirling shire; Alexander Napier of Gillets, Esq; William Napier of Ardmore; and Adam, of whom the Napiers of Blackstone and Craigannet in Stirling shire are descended. His daughters were, Margaret, the wife of Sir James Stuart of Roffayth; Jane, married to James Hamilton, Laird of Kilbrachmont in Fife; Elizabeth, to William Cunningham of Craigends; Agnes, to George Drummond of Baloch; and Helen, to The

Reverend

* In the University Library at Edinburgh, another in the possession of the Lord Napier, and an 8vo print, engraved by Delaram, where he is represented calculating with his bones.

† Robert wrote a Chemical Treatise, still preserved in the family of Napier.

Reverend Mr Mathew Busbane, Rector of the Parish of Erskine in Renfrew shire.

HE was interred in the Cathedral Church of St Giles, at Edinburgh, on the east side of its northern entrance, where there is now a Stone Tablet, indicating, by a Latin Inscription, that the burial place of the Napiers, is in that place ; but no Tomb has ever been erected to the memory of so celebrated a man, nor can it be required to preserve his memory, since the astronomer, geographer, navigator and political arithmetician, must feel themselves every day indebted to his inventions, and thus a monument is erected to the illustrious Napier, which cannot be obliterated by time, or deprettiated by the ingenuity of others in the same department.

I proceed now to evince more fully the merit of Napier, by giving an account of the state in which he found Arithmetic, and of the benefit it received from his discoveries.

SECTION

S E C T I O N I.

CONCERNING ARITHMETIC.

An cum Statuas et Imagines, non animorum simulacra sed corporum, studiose multi summi homines reliquerunt ; confiliorum relinquere, ac virtutum nostrarum effigiem nonne multo malle debemus, summis ingenii expressam et politam ?

CICERONIS ORATIO PRO ARCHIA POETA. CAP. XII.

ARITHMETIC is so necessary to man, that it must have made its appearance on the first and rudest stage of society.

SIGNS to express numbers were probably in use, as soon as signs to express other ideas.

THE signs the most obvious, and we may venture to say the first in use, were the fingers. The number of these accounts for the general adoption of numeration by tens. The first ten numbers have the appellation of digits or fingers, in most of the languages.

THE next improvement of Arithmetic, seems to have been the use of small pebbles, or of knotted strings. The words *λιθαγον* and *calculus*

F

signify

signify both a pebble and an arithmetical operation. The Russians, to this day, perform their calculations by means of stringed beads, with great exactness and expedition. The Greeks and Romans represented numbers by the letters of the Alphabet variously combined. By means of their notation, the operations of addition and subtraction of integers at least, were easily enough performed. But multiplication, division, and the extraction of roots, were difficult and tedious operations. They must have effected them, in a great measure, by dint of thought. Boethius, who flourished towards the end of the fourth century, says indeed, that some of the Pythagoreans had invented, and used in their calculations, nine apices or characters, resembling those we now employ; by which these latter operations must have been much simplified. These figures were known only to a few mysterious men, and it is by no means probable that they were the inventors of them. It is probable that Pythagoras, or some of his disciples, borrowed them, as they did many other inventions, from the Indians. The merit of the Greek Philosophers, of which Euclid claims a distinguished share, consisted in raising Arithmetic, from being a simple art, to the rank of the sciences.

GERBERT of Aquitaine, in France, afterwards Pope Sylvester the second, having imbibed the elements of the sciences, found that the Christian world, at that time involved in darkness, could not furnish him with sufficient helps for making any great progress in them. This induced him to fly from the Convent of Fleury, where he had lived from his infancy, to Spain; where, under the tuition of the Moors, he became so intimately acquainted with the mathematics, that he is said to have soon surpassed his masters. Upon his return to his native coun-

try,

try, about the year 960, or 970, he introduced the ten characters, which form the basis of our modern Arithmetic. These had been familiar to the Arabs, time out of mind, and the invention of them is, by their writers, ascribed to the Indians *.

ABOUT five hundred years afterwards, our Arithmetic received a most important improvement, by the invention of decimal fractions.

As the invention of these fractions, and of the Logarithms, with other arithmetical improvements, was occasioned by the efforts of ingenious men, to perfect Trigonometry, it will be proper to give some account of the rise and progress of this most useful branch of the mathematics.

TRIGONOMETRY, considered as a simple art, must have begun with the division of lands in every country; but considered as a science, or as the application of Arithmetic to Geometry, it seems to have had its rise among the hands of the great Hipparchus, about one hundred and forty years before the christian æra. Hipparchus was the first who made use of the longitudes and latitudes, for determining the position of places, on the surface of the earth. Theon cites a treatise of his, in twelve books, on the chords of circular arcs, which must have been a treatise on Trigonometry, and is the first of which history gives any account. Menelaus, about the end of the first century, wrote a treatise, in six books, on the chords; and there are extant of his three books on Spherical Trigonometry, where that subject is treated in a manner very profound and extensive.

THE

* Wallis, Montucla, &c.

THE difficulties to be encountered in the solution of triangles, which is the object of Trigonometry, regard the tables of the parts of the circle, the form of the problems to be used, and the application of these problems to practice.

THE Ancients, before Ptolemy's time, do not seem to have agreed upon a particular division of the radius of the circle *. That indefatigable Astronomer, who flourished about the year 200, having simplified the theory of Menelaus, divided the radius into sixty equal parts, and computed on this foundation, the length of all the chords in the semi-circle, corresponding to every thirty minutes. This sexagenary division, which continued in use for many centuries, obliged geometers to make use of numbers composed of integers and fractions, which occasioned much labour and much loss of time. The table of chords led them to problems very complicated and of difficult execution. Every oblique triangle was to be divided into two rectangular ones; and in order to come at a solution, it was necessary to raise to the square, and to extract the square root of many fractional numbers.

THE Arabs, sometime in the eleventh century, greatly simplified the theory of Trigonometry, by substituting, for the chords of the double arcs, the halves of these chords. These lines have been called *sinus*, probably from S. Ins. an abbreviation of the Latin words *semisses inscriptarum* †. This improvement paved the way to more simple theorems, of which we shall have occasion afterwards to speak.

ABOUT

* Hanschii Pref. in Kepl. Epist. † Montucla Histoire Mathematique.

ABOUT the middle of the fifteenth century, George, furnamed Peurbach, from a village on the confines of Austria and Bavaria, where he was born, either adopted the sinus from the Arabs, or invented them himself. He also banished from Trigonometry the use of the sexagesinary calculus, by supposing the radius to consist of 600 000 equal parts, and computing on this foundation the length of the fines corresponding to every ten minutes of the Quadrant.

JOHN Muller (commonly known by the name of Regiomontanus from the place of his birth, Konigsberg a town in Franconia) the disciple of Peurbach, improved his master's idea by making the radius equal to unity or 1,0000000. On this new plan he calculated, with great labour and accuracy, a table of the fines for all the minutes of the Quadrant. He also was the first who introduced the use of the tangents in Trigonometry; of which Erasmus Reinoldus of Salfeldt first constructed a table. To these tables Rheticus * afterwards added that of the secants, which had been invented by F. Maurolyeus of Messina.

By means of these new tables the art of Trigonometry was not only rendered more accurate than formerly, but one multiplication or division was superseded in every geometrical proportion where the radius made one of the terms. The multiplication or division, however, of such large numbers required much expence of time, labour and attention.

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RAYMARUS

* George Joachim, so called from his native country. These appellatives, so much used after the revival of letters, make it often difficult to discover the real names of learned men.

RAYMARUS Ursus, towards the end of the sixteenth century, having either learned from his preceptor Justus Byrgius, or discovered some new properties of the sines, shewed, in his *Fundamentum Astronomicum* published in the year 1588, how these might be employed to great advantage in the solution of some trigonometrical questions. By his method, which he calls *Prosthaphæresis*, from *προσθίσσειν additio* and *ἀφαιρέσσειν ablatio*, the fourth term of a geometrical proposition, having for its first term the radius equal to unity, may be found by addition and subtraction only; instead, for example, of multiplying the sine a by the sine of b in the geometrical proportion $1 : \sin. a :: \sin. b : \sin. c$, the sine of c may be had, with much less trouble, by subtracting half the cosine of the sum of a and b from half the cosine of their difference; because, as is easily demonstrated, $\sin. a \times \sin. b = \frac{1}{2} \cosin (a+b) - \frac{1}{2} \cosin (a-b)$.

IT was only to a few cases, however, that the prosthaphæresis of Raymar could be applied, and the improvements made upon it, by Clavius * Magini † and others, required so many precautions that they were not of very great service. ‡ But inconsiderable as these abbreviations of calculus were, they were generally used by the most eminent mathematicians and astronomers at the end of the sixteenth and beginning of the seventeenth century §.

SUCH

* Clavius de Astrolabio, book 1. lemma 53.

† Magini primum mobili, lib. 1. theor. 33. and lib. 11. cap. 2.

‡ Quod vero Prosthaphæresis tabulis attinet, scito me totum hunc annum qua parte et a morbis et a curis fui vacuus in unius martis prosthaphæresibus excentri versari, nec pudet dicere me scopum bonum attigesse. Kepler Epist. p. 171.

§ Prosthaphæretical tables were published by J. G. Herwart, in 1610.

SUCH was the state of arithmetical computation, at the time of the invention of the Logarithms, which, as Napier himself says, *Omnem illam pristinæ matheſeos difficultatem penitas e medio tollit; et ad sublevandam memoriam imbecillitatem ita ſe accomodat, ut illius adminiculo facile ſit, plures quæſtiones mathematicas unius horæ ſpatio, quam priftinia et communiter recepta forma ſinuum, tangentium et ſecantium, vel integrō die abſolvere**. But before we proceed to this most important discovery, we ſhall give an account of those ingenious contrivances, intended to answer the ſame purpose, which previously occurred to Napier.

SECTION II.

NAPIER'S BONES.

THE first of these mechanical devices is what our author calls Rabdologia, or the art of computing by figured rods. These rods are square parallelepipeds three inches in length, and three tenths in breadth. Each of the faces of these parallelepipeds is divided into ten equal parts, of which nine are squares and in the middle, and half of the tenth at one extremity or the top, and half at the other extremity or the bottom. Every one of these squares is cut by a diagonal from left to right upwards. At the top of each face is ſome one of the ten digits 0, 1, 2, 3, &c.

IN

* Log. Canon. descriptio. in dedic.

IN the first square below that digit is repeated, in the second is its double, in the third it's triple, and so on. Of these multiples of the digit, the figure of units is below, and the figure of tens above the diagonal. The meaning of what has been just said will be evident by a little attention to Fig. I. where the four faces of each rod of the set, recommended by Napier, are unfolded. By means of these rods the operations of multiplication and division are performed by addition and subtraction.

THE rule for multiplication is—Bring the rods to form the multiplicand at the top of their upper face. Join a rod, having unity at the top of its upper face, to the right or left hand side; in which seek the right hand figure of the multiplicator, and write out the numbers corresponding thereto in the square of the other faces, by adding the several numbers occurring in the same rhomboid. Seek the second figure of the multiplicator and proceed in the same manner: arrange and add the numbers wrote out, as in common multiplication; the sum is the product required. To multiply 1785 by 364, for example, I dispose the proper rods as in Fig. II.; next to 4 (the first right hand figure of the multiplicator) I find 0; in the contiguous rhomboid 2 and 2, which added together make 4; in the next 3 and 8 which make 1 and a surplus of ten; and in the last 2 and 4 which, together with unity for the ten I had in the former rhomboid, make 7. These numbers 0, 4, 1, 7, I set down one after the other as I find them, proceeding from right to left. I go on in the same manner with 6 and 3 (the other figures of the multiplicator); and, after arranging and adding the partial products I find the total product required. Thus,

$$\begin{array}{r}
 364 \\
 \hline
 7140 \\
 10710 \\
 5355 \\
 \hline
 649740
 \end{array}$$

The rule for division. BRING the rods to form the divisor at the top of their upper face. Join a rod having unity at the top of its upper face, to the right or left hand side. Descend under the divisor till you meet those figures of the dividend wherein it is first required how often the divisor is found, or the next less number, which subtract from the first figures of the dividend, and put for the first figure of the quotient the corresponding number on the side face. Bring down, one after the other, the remaining figures of your dividend as in common division, and proceed in the same manner till you have finished the operation. Let it, for example, be required to divide 649740 by 364. I dispose the rods as in Fig. III. The next less number under the divisor 364 to 649 (the first figures of the dividend) I find to be 364 itself which I subtract from 649 putting 1, the number corresponding on the side face, for the first figure of my quotient: to my remainder 285 I bring down 7 the next figure of my dividend. The next less number to 2857 under the divisor I find to be 2548, which I subtract from 2857, putting 7, the number corresponding in the side face, for the second figure of the quotient. I go on in the same manner till I have brought down the other figures of the dividend and completed my quotient as follows.

649740 (1785

364

2857

2548

3094

2912

1820

1820

(0)

ALTHOUGH the extraction of the square and cube roots may be pretty expeditiously performed by the rods, Napier proposes an auxiliary lamella for the abridgement of it. It would serve little purpose to give a particular description of the lamella, or an account of the manner of using it. Its length and thickness are the same with those of the rods, and its breadth quadruple. Its two faces are divided and marked as in Fig. IV. To find out the way of operating by it will be no difficulty to any body who is a little acquainted with arithmetic and has time to spare.

ANOTHER of Napier's contrivances is his *multiplicationis promptuarium*.

THIS machine consists of a box of figured lamellæ. The lamellæ, two hundred in number, are each eleven inches in length and one inch in breadth. Each of these lamellæ is divided into eleven equal parts of which ten in the middle are squares, and two thirds of the eleventh at one

one extremity, and one third at the other. Every one of these squares is divided into nine less squares, one hundred of the lamellæ are each one fourth of an inch in thickness, and the other hundred one eighth. Suppose the former, which we shall call direct lamellæ, placed so that the greater margin may appear at top and the less at bottom; and the latter which we shall call transverse, placed laterally, with the greater margin to the right and the less to the left hand. In this position every square appears cut by a diagonal (faint in the small but strong in the great ones) from the left to right upwards. Each of every ten both of the direct and of the transverse lamellæ has some one of the ten digits $0, 1, 2, 3, \&c.$ inscribed on its greater margin. The multiples of the digit on the margin of a direct lamella are disposed in each of its greater squares as pointed out by Fig. V. where a represents the digit itself, b the right hand figure, and b' the left hand figure of its double; c and c' the right and left hand figures of its triple (the plain letters being above and the accented ones below the diagonal of the figure); d and d' those of its quadruple, and so on. In the transverse lamellæ those which have 0 on the margin are untouched; those which have unity on the margin have the loculus corresponding to a cut out; those which have two on the margin have the loculi of 6 and $6'$ cut out; those which have 3 the loculi of c and c' ; those which have 4 the loculi of d and d' , &c. This will be sufficiently evident by inspecting Fig. VI. where it is exemplified in a direct lamella titled with the digit 4 , and in a transverse one with 7 . The box fitted to receive these lamellæ is of a cubical form; something more than eleven inches wide and nearly eight inches high; see Fig. VII. Two of its contiguous sides, which we shall distinguish by the names of left and right, are divided

divided into twenty parts, each equal in length to the breadth of ten lamellæ, and in height to the thickness of a direct and of a transverse lamella alternately. The greater divisions on the left side are cut out, and the less on the right side. Into the box through each of the former, with their titled ends foremost, ten direct lamellæ of the same title are inserted with their untitled ends foremost, and an equal number of the transverse ones of the same title, through each of the latter. Those titled *o* are in the uppermost divisions, and those titled *1, 2, 3, &c.* in the respective divisions below.

Multiplication by the promptuary is performed as follows. THE first, or right hand, second, third, &c. figure of the multiplicand is exhibited by the title of a lamella taken from the first, or right hand, second, third, &c. column of the left side of the box and placed on its lid exactly above and as it lay in that column. The empty space, if any, towards the left is to be covered with blank lamellæ. The first, or right hand, second, third, &c. figure of the multiplicator is exhibited by the title of a lamella taken from the first, or left hand, second, third, &c. column of the right side of the box and placed on the former lamella exactly above, and as it lay in that column. The remaining spaces, if any, towards the right are to be covered with blank lamellæ. All the useful multiples on the direct lamellæ appear through the *feneſtelle*, and all the useless multiples are hid. All the numbers beginning at the corner next the first or right hand figures of the multiplicand and multiplicator, lying between the united strong diagonals, are to be added severally; the right hand figures of the sums, marked down; and *1* for every *10*, carried to the next place, till we come to the opposite

site

site corner: and the work is done. This operation, we trust, is described with sufficient accuracy and plainness to supersede the necessity of an example. In order that division may be performed by the Promptuary, it must first be converted into multiplication by means of tables dressed on purpose, or of tables of the sines, tangents and secants, constructed on the hypothesis of the radius being equal to unity, followed by a certain number of Zeros. That this may be accomplished by these last, look for the co-secant, or co-tangent of the arc which has the divisor for its sine or tangent. Make the co-secant or co-tangent found the multiplicand, and the dividend the multiplicator; or conversely. Find the product by the promptuary as above directed. This product, a number of the right hand figures according to the number of zeros in the square of the radius being marked off as decimals, is the quotient required. The reason of this is obvious: the co-secants or co-tangents being third proportionals to the sines or tangents and the radius or unity; to multiply any number by one of the two first, or to divide it by the corresponding one of the two second of these lines, is one and the same thing.

LOCAL ARITHMETIC.

LOCAL Arithmetic, another ingenious invention of Napier, is the art of calculating by means of counters properly placed on a chess board, or similar table. Two contiguous margins (which we shall distinguish by the names of left or inferior, and right or lateral) of that table, are divided into a number of parts equal to that of their adjoining squares. The inferior divisions beginning at the right and the lateral at the left have successively inscribed in them the successive terms of the geometrical progression 1, 2, 4, 8, 16, 32, &c. which are called local numbers.

COMMON numbers are reduced to local by subtraction, and local numbers to common by addition. The common number 1875, for example, expressed in local numbers will be found to be 1024; 512; 256; 64; 16; 2 and 1: and *vice versa*. The first of these reductions is necessary before, and the second after any of the operations of common arithmetic are performed by this contrivance. By the help of a very simple table, reduction may be performed with ease and expedition.

To Add. PUT a counter for each local number in the proper place on the lateral or on the inferior margin of your table. For every two counters found in the same place, put one in the next higher, after removing them. Repeat this till no place shall contain more than one counter. The counters left indicate the numbers required. Thus let it be required to find the sum 1875; 258, and 1099. I put the coun-

ters

ters at 1024; 512; 256; 64; 16; 2 and 1, the local numbers of the first; at 256, and 2, those of the second; and at 1024; 64, 8, 2, and 1, those of the third. At 1.—I find two counters which I remove, and put a counter at 2 where I find other three. I take up these four and put two, in the next place 4 &c. and proceeding in this manner I find at last a counter at each of the following numbers 2048; 1024; 128, and 32, which form 3232 the sum sought.

To Subtract. PUT a counter for each local number of the greater of the two quantities, at its proper place, a little to one side, on the inferior margin; and one for each of the local numbers of the less of the two quantities, at its proper place, a little to the other side, on the same margin. Remove the counters found on opposite sides of the same place. Change the side of the lowest counter remaining; take off that above it; and put a counter in each place between them. Remove as before. Repeat this till there shall be no counters on one of the sides of the margin; and those on the other will indicate the remainder. Let it be proposed, for example, to subtract 1099 from 1875. I put counters at 1024; 64; 8; 2 and 1, to the left of the lateral margin, and at 1024; 512; 256; 64; 16; 2 and 1, to the slit of that margin. Finding a counter on each side of the numbers 1024; 64; 2, and 1, I remove them. My lowest counter is to the left of 8. I put it to the right and take up 16. above it; as there are no intermediate places, and as the remaining counters are on the same side of the margin my operation is finished. The remainder is 512; 256, and 8; or 776.

MULTIPLICATION, Division, and the extraction of the square root, may also be performed on the margin: but they are performed with much greater ease and clearness on two contiguous margins and the squares of the table. On these last the counters have two different movements; the one parallel to the sides like that of the tour, and the other diagonal like that of the bishop, on the chess board. Every square of the table is said to have for its value one of the equal numbers (on the two margins) between which it lies diagonally. The two sides of a square formed by counters in the area of the table, parallel to the inferior and lateral margins, we shall call a Gnomon: this gnomon consisting of 3, 5, 7, &c. counters is said to be congruous when its value can be subtracted from the numbers left marked upon the margin.

To Multiply. MARK with counters the local multiplicator in the inferior and the local multiplicator in the lateral margin. From the middle of the marked places let points be supposed to move perpendicularly into the table, and put a counter at each intersection. Remove the counters on the margins. Bring the counters in the squares of the tables to their values in one of the margins; add, if necessary, and the work is done. Suppose, for example, 19 is to be multiplied by 13. I mark with counters Fig. VIII. the numbers 1, 2, and 16, on the inferior and the numbers 1, 4, and 8, on the lateral margin, having placed other counters rectangularly in the table, I remove the marginal ones. Those other counters I bring up, one by one, to their proper places in the lateral margin; and, after adding, I find a counter at each

of

of the following numbers, 128; 64; 32; 16; 4; 2, and 1, which form my product, 247.

To Divide. MARK with counters the local dividend in the lateral, and the local divisor in the inferior margin, beginning at the square where a point, descending diagonally from the angle above the highest number of the dividend, would intersect a point ascending perpendicularly from the highest number of the divisor; place a series of counters parallel to the divisor: If this series is equal or inferior in value to the higher number of the dividend subtract it from them; and if otherwise, bring it down one, two, &c. steps and subtract. Repeat the operation till either nothing, or at least a numberless than the divisor, shall remain on the lateral margin. These serieses point to the numbers that form the quotient. For example let it be required to divide 250 by 13. I mark, Fig. IX. the numbers 128; 64; 32; 16; 8; and 2, in the lateral, and 8; 4, and 1 in the inferior margin.

My first series points to 16. I subtract it from the dividend and find remaining 32; 8, and 2.

My next series pointing to 4 is too great to be subtracted, for which reason I bring it a step farther down.

AFTER subtracting, there remains 16. In the same manner my third series pointing to 2 I must bring to point to 1; which subtracted, there remains counters on the dividend at 2 and 1. My quotient is therefore 16; 2, and 1, or 19; and 2 and 1, or 3 over.

To extract the square root. MARK the number locally in the lateral margin. From the angle formed by the meeting of the inferior line with the lateral, let a point ascend diagonally till it arrive in a square of the same value with the highest number that can be subtracted from the number whose square root is sought. In this square place a counter, and subtract its value from the number marked in the margin. Form the congruous gnomons, which from the foreaid square have each their upper counter in a line perpendicular and their left hand inferior one in a line parallel to the lateral margin: and subtract their value one by one from the marked remainders. The counters, lying perpendicularly to either of the margins, point out the square root. Let it be proposed, for example, to find the square root of 2209; I mark the numbers 2048, 128, 32, and 1 on the lateral margin. Fig. X. Subtracting the value 1024 of the first counter placed in the table as directed, the remainders are 1024, 128, 32 and 1. From these taking the value 512 and 64 of the first congruous gnomon, there remain 512, 64, 32 and 1. From these taking the value of the second congruous gnomon 256, 64 and 16, there remain 64, 16, 8, 4 and 1: and from these taking the value of the fourth congruous gnomon, 64, 16, 8, 4 and 1, there remains nothing. The square root, as indicated by the direction of the counters in the table, is 32, 8, 4, 2 and 1, or 47.

WHAT is above said will, it is hoped, be sufficient to give a clear idea of the form and use of those of Napier's arithmetical instruments, which seemed to him worthy of being communicated to the public. The reasons on which the different operations are founded, depending upon the construction of the machines and the obvious properties of numbers,

bers, must occur to every reader endowed with a moderate share of attention. The hint of the rods, or *virgulæ numeratrices* and of the promptuary which is only an improvement of the rods, seems to have been taken from the *Abacus Pythagoricus* or common multiplication table. Napier's acquaintance with chess, the most ingenious of all games, and at which mathematicians are commonly the best players, occasioned his discovery of the *Arithmetica localis*. The Promptuary, at least for multiplication, is greatly preferable to the rods and the chess board; for the partial products of two numbers, consisting of even ten Figures each, may, by a little practice, be exhibited on that machine in the space of a minute, and no numbers require to be written out, excepting the total product. Had the logarithms remained undiscovered, the promptuary, in all probability, would have become universally familiar to those who were engaged in tedious calculations. But to those who are acquainted with the logarithms, Napier's arithmetical machines and those afterwards invented, a few of which we shall enumerate, although the monuments of genius, must, in general, be regarded as mathematical curiosities of no use.

PERHAPS put into the hands of young people learning arithmetic, they might make them fond of that study.

SHICKARTUS in a letter to Kepler, written in the year 1623, informs him that he had lately constructed a machine consisting of eleven entire and six mutilated little wheels, by which he performed the four arithmetical operations*, Pascal, in the year 1642, at the age of nine-

teen,

* Kepl. Epist. p. 683.

teen, invented a machine with which all kinds of computations may be made without the pen, without counters, and without the knowledge of any rule of arithmetic. I have not been able to meet with any description of it. It must however have been of a very complicated nature as its author was two years in bringing it to perfection, owing to the difficulty he found to make the workmen understand him thoroughly *. The French writers agree in calling it admirable ; † but the name of Pascal perhaps does it more honour than it deserves. This machine is preserved in the cabinet of the king of France and in those of a few others ‡.

THE Marquis of Worcester, a man of genius but a plagiary, mentions in his scantlings of inventions, published in the year 1655, an instrument whereby persons ignorant in arithmetic may perfectly observe numerations and subtractions of all sums and fractions §. Whether he here refers to some of Napier's instruments, to Gunter's scale, of which I shall afterwards speak, or to some invention of his own is uncertain.

ABOUT the year 1670, ** Sir Samuel Moreland contrived two arithmetical instruments ; one for addition and subtraction, and the other for multiplicaton, division, and the extraction of the square, cube, and square cube roots, the description of which he published at London, anno 1671 ††.

MUCH

* Les hommes illustres par Perrault vie de Pascal. † Bayle Chauffepie, Baillet, Perrault, &c.

‡ Pref. Pensées de Pascal. § N° 84. Glag. Edit. 1767.

** Moreland's Instrument of excellent use as well at sea as at land, invented and variously expen-
mented in the year 1670, Lond. 1671. Fol.

†† See also Phil. Transact. N° 94. p. 6048.

MUCH about the same time, Mr George Brown, afterwards Minister of Kilmaures in Scotland, invented a machine which, in his account of it published at Edinburgh in the year 1700, he calls the *Rotula Arithmetica*. This machine consists of a box containing a circular plane moveable on a center pin and fixed ring, whose circles are described from the same center. The outermost circular band of the moveable, and the innermost of the fixed, are each divided into a hundred equal parts, and these parts are numbered 0, 1, 2, 3, &c. Upon the ring there is a small circle having its circumference divided into ten equal parts, furnished with a needle which shifts one part at every revolution of the moveable. By this simple instrument are performed the four arithmetical operations not only of integers but even of decimals finite and infinite *.

SOME time before Mr Brown's little book appeared, Mr Glover had published a *Roue Arithmetique* similar to the Rotula but not so perfect. It would appear however that that gentleman had got some hints of the construction of the Rotula from a brother of his own who had been one of Brown's pupils in 1674 †.

IN the year 1725, an instrument invented by M. de l'Epine of a more simple construction and easier in its operations than Pascal's; in 1730, another by Mr Boissendeau, by which calculation is performed without writing; and in 1738 a third by Mr Rauslin, consisting of rods different from those of Napier, were approved of by the French academy ‡.

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SAM

* One of these machines is in the library belonging to the faculty of advocates at Edinburgh.

† Rot. Arithm. Pref. ‡ See the Paris memoirs for these years. Histoire.

SAM Reyer invented, at what time I have not been able to learn, a kind of sexagenal rods in imitation of Napier's, by which sexagenary arithmetic is easily performed *.

I have an arithmetical machine which came into my possession from my uncle George Lewis Erskine who, though born deaf, by the assistance of the learned Henry Baker of the Royal Society at London, acquired not only the use of speech and the learned languages but a deep acquaintance with useful literature. This machine consists of a small square box furnished with six cylinders moveable round their axes. Upon each of these cylinders, which are only Napier's rods, are engraven the ten digits, and their multiples. From a perpetual almanac on the out side of the box, it would appear that this machine was constructed in the year 1679.

SECTION

*See Chamber's Diction. Article Arithmetic.

S E C T I O N III.

NAPIER'S THEORY OF THE LOGARITHMS*: NEWTON'S IDEAS OF FLUXIONS, BORROWED FROM NAPIER.

I Shall now proceed to unfold the Logarithms, the discovery of which has justly entitled Napier to the name of the *greatest Mathematician of his Country*. Let two points, the one in N, and the other in L, (Fig. XI.) having at first a similar velocity, move along the indefinite straight lines CND and KL Δ ; the first increasing its velocity or diminishing it according to its distance from a fixed point C, and the second preserving its velocity without augmentation or diminution. Let the former, in a certain time, arrive at any point N' or n', and the latter in the same time at the point L' or l': the space L L' or L l' described by the second moveable point is said to be the Logarithm of the distance C N' or C n' of the first from the fixed point C.

1. THE Logarithm of C N or unity is zero: for the first moveable point not having left N, the second has had no time to describe any space.

2.

* The term Logarithm was first used by Napier after the publication of the Canon in which he uses the term of *numerus artificialis*.

2. THE Logarithms of the terms of a geometrical series are in arithmetical progression: for let NN' , NN'' , $N''N'''$, &c. or Nn' , $n'n''$, $n''n'''$, &c. be continual proportionals, they will be described by the first moveable in equal times, and the equal spaces LL' , $L'L''$, $L''L'''$, &c. or Ll' , $l'l''$, $l''l'''$, &c. will be described by the second moveable in the same times. Now it is easily demonstrated that CN , CN' , CN'' , &c. or Cn , Cn' , Cn'' , &c. are in geometrical progression, and it is evident that their respective logarithms o , LL' , $L'L''$, &c. or o , Ll' , $2LL'$, &c. and o , Ll' , Ll'' , &c. or o , Ll' , $2Ll'$, &c. are in arithmetical progression.

3. THE logarithms of quantities less than CN are negative, if those of quantities greater than CN are positive; and conversely: for if Cn' , CN , CN' , CN'' are continual proportionals, in order that their logarithms $2Ll'$, Ll' , o , LL' , $2LL'$, &c. may be in arithmetical progression it is necessary that the terms on different sides of zero should have opposite signs. Hence,

4. THE logarithm of any quantity is the same with that of its reciprocal, the sign excepted.

5. THE number of systems of logarithms is infinite: for the ratio of CN to CN' and LL' are indeterminate.

6. THE logarithms of any one system, are to the correspondent ones of any other, as the value of LL' in the first system, is to its value in the

the second. From the 2d proposition the four following, expressed in the language of arithmetic, are easily deduced.

7. THE logarithm of a product is equal to the sum of the logarithms of its factors. Thus the logarithm of $CN' \times CN''$ is $LL' + LL'' = LL''$: for $CN' \times CN'' = CN'''$.

8. THE logarithm of a quotient is equal to the difference of the logarithms of the divisor and dividend. Thus the logarithm of $\frac{CN''}{CN'}$ is $LL'' - LL' = LL'$: for $\frac{CN''}{CN'} = CN''$.

9. THE logarithm of the power of a quantity is equal to the product of the logarithm of that quantity by the index of its power. Thus the logarithm of $\sqrt[3]{CN'}$ is $3 LL' = LL''$: for $\sqrt[3]{CN'} = CN'''$.

10. THE logarithm of the root of a quantity is equal to the quotient of the logarithm of that quantity by the index of its root. Thus the logarithm of $\sqrt[3]{CN''}$ is $\frac{1}{3} LL' = LL'$: for $\sqrt[3]{CN''} = CN'$.

FROM the 7th and 8th propositions the two following are evident.

11. THE logarithm of an extreme or mean term of a geometrical proportion, is equal to the difference of the sum of the logarithms of the means or extremes and the logarithm of the other extreme or mean.

12. If the logarithms of all the primary numbers are known, those of all the composite numbers may be found by simple addition; and if all the latter are known, all the former may be known by simple subtraction.

FROM the 2nd or the 9th and 10th propositions.

13. The logarithms may be thus defined, *Numerorum proportionalium æquidifferentes comites*; or more properly (as their name, *λογαριθμός*, imports) *Numeri rationem exponentes*; because they denote the rank, order, or distance, with regard to unity, of every number in a series of continued proportionals of an indefinite number of terms.

14. THE logarithm Ll' of any quantity Cn' is greater than the difference Nn' between CN or unity and that quantity, and less than that difference, increased in the proportion of CN to the said quantity: for the velocity of the second moveable describing Ll' being greater than that of the first describing Nn' during the same time, Ll' is greater than Nn' or $CN - C'$; and the velocity with which NN' is described, being greater than that with which Ll' is described, in an equal time, Ll' is less than NN' or $CN' - CN$ or [since $Cn': CN :: CN': CN'$], $[CN - Cn'] \times \frac{CN}{Cn'}$

Hence,

15. If a quantity Cn' differs infinitely little from CN or unity, its logarithm Ll' will be equal to $\frac{[CN + Cn'] \times [CN - Cn']}{2 Cn'}$ the arithmetical,

metical, or to $[CN - Cn] \times \sqrt{\frac{CN}{Cn}}$ the geometrical mean between its limits above stated.

16. THE difference ll'' of the logarithms Ll' and Ll'' of any two quantities Cn' and Cn'' is less than the difference $n' n''$ of these quantities increased in the proportion of the lesser Cn'' to CN or unity; and greater than the said difference increased in the proportion of the greater Cn' to CN or unity: for reasoning in the same manner as in the 14th proposition ll'' will be found to be less than NN' or [since $Cn'':CN :: n' n'':NN'] $CN \times n' n''$. Hence,$

17. If the difference of two numbers Cn' and Cn'' is infinitely small, the difference of their logarithms will be expressed by the arithmetical $(Cn' + Cn'') \times (Cn' - Cn'') \times CN$ or the geometrical mean $\frac{Cn' - Cn''}{\sqrt{Cn' \times Cn''}}$

$\times CN$ between its limits above stated. Beautiful, ingenious and profound! Such is the manner in which Napier conceived the generation of numbers and their logarithms, and such are some of their relative properties which naturally flow from it. Those who are acquainted with Newton's manner of explaining the doctrine of fluxions, must be struck at its resemblance to this of our Scottish Geometer. This resemblance, or rather identity, is conspicuous not only in their ideas but in their very words. The explanation of the first definition in the *Canonis mirifici descriptio* is in the following terms: *Sit punctus A, a quo ducenda sit linea fluxu alterius puncti, qui sit B. Fluat ergo primo momento B ab A*

in

in C. Secundo momento a' C in D. Tertio momento a' D in E atque ita deinceps in infinitum describendo lineam A C D E F, &c. Intervallis AC, CD, DE, EF et ceteris deinceps aequalibus, et momentis aequalibus descriptis, &c. In the appendix to the *Canonis mirifici constructio*, under the article *Habitudines Logarithmorum*, he thus expresses the relation between two natural numbers and the velocities of the increments or decrements of their logarithms; *Ut sinus major ad minorem; ita velocitas incrementi aut decrementi apud majorem*. What difference is there betwixt this language and that of the great Newton now in use $x:y::\text{Log. } y:\text{Log. } x$ *?

THE seeds of the invention of the logarithms were perceived by the ancients as well as by the moderns, upon the revival of science in Europe, before the time of Napier. In the elements of Euclid, and in the Arenarius of Archimedes †, these great men seem to have been very well acquainted

* See Hutton's Construction of Logarithms, p. 42 and 48.

†. [In the Arenarius of Archimedes] Without entering in this place, on the repulsion of the received opinion, that this great Mathematician had made the first step towards the knowledge of the Logarithme, I shall content myself with giving the result of the enquiry, by one of the ablest Mathematicians in the country, to whom I addressed myself, when I first set myself to produce this work, and who having successfully illustrated the discoveries of the Prince of English Mathematicians, gladly came forward to contribute his share to the triumph of our Scotish Newton.

Archimedes demonstrates a Theorem concerning numbers, made by the mutual multiplication of the terms of a geometrical progression; by means of which Theorem the principles of Logarithmic computation may easily be demonstrated. Archimedes, therefore, had he been furnished with tables of Logarithms, would have known how to use them: But it appears not, that he was possessed of any principles, which could lead him to the formation of Logarithms. He could avail himself, indeed, of the indices of the powers of numbers, to abridge the labour of multiplication, as we now avail ourselves of Logarithms for the like purpose: But the gulph between this method by the *Natural Indices*:

acquainted with the correspondence of an arithmetical to a geometrical progression *.

MICHAEL Stifellius, a German Arithmetician, who flourished about the middle of the sixteenth century, in his *Arithmetica integra* stated the comparison between the series $\{1, 2, 4, 8, 16, 32, \dots\}$ &c. observing that the product or quotient of any two terms of the former corresponded to the sum, or difference, of the equidistant terms of the latter.

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indices, and the method of Logarithms, is wider than it may at first seem. Any Number, not itself arising from a root, is the root of a distinct progression of Powers. Hence there are as many distinct progressions as there are numbers not actually powers: And in all these progressions the homologous powers have the same exponents or indices. Thus 3 is the exponent of the number 8 in the series of the powers of 2. But 3 is equally the exponent of 27 in the series of the powers of 3; of 125 in the series of the powers of 5; of 343 in the series of the powers of 7; and universally of all cubic numbers; so 4 is the exponent of all biquadratic numbers; 5 of all quadrato-cubic; and so on. A number therefore is not sufficiently characterised by its exponent unless it be known to what series of powers it belongs, that is from what root it arises. Add to this that many numbers fall into no natural series of powers. This method therefore of computing by the natural indices of powers arising from the natural numbers as roots, will only serve the purpose of rude calculations leading to some very general conclusions, and must fail in all instances in which accuracy is required. Archimedes never thought of considering all numbers as expressions of proportions, capable of being universally included in one general series of ratios, which notion is the true basis of Napier's great invention, as will be more fully explained hereafter. For the invention in effect was this; that he found a method to raise a series of proportionals, in which all numbers should be comprised, in which every number of consequence had its own particular exponent, and to find the exponent of any given number, or the number of any given exponent in that universal series."

In the course of this work, it will be sufficiently proved, that Napier was as much the first to conceive as to execute this wonderful project.

* Those who wish to recollect how much we are indebted to the ancients, in this as well as in many other departments of sciences, will read with pleasure Mr Duten's Inquiry into the origin of the discoveries attributed to the moderns.

WHETHER Napier ever saw or heard of this remark of Stifellius is not known, nor indeed is it of any consequence; for it cannot fail of presenting itself to any person of moderate acuteness who happens to be engaged in arithmetical questions of this nature where the powers of numbers are concerned. It is not therefore this barren though fundamental remark that can entitle him who first mentioned it to the name of the inventor of the logarithms. The superior merit of Napier consists in having imagined and assigned a logarithm to any number whatever, by supposing that logarithm to be one of the terms of an infinite arithmetical progression, and that number one of the terms of an infinite geometrical progression whose consecutive terms differ infinitely little from each other.

THE invention of the logarithms has been attributed to Christianus Longomontanus, one of Tycho Brahe's disciples, and an eminent astronomer and mathematician in Denmark. The hackneyed story which gave rise to this, is told by Anthony Wood in his *Atbenæ Oxonienses**, and is as follows; "One Doctor Craig, a Scotchman, coming out of Denmark into his own country, called upon John Neper baron of Merchiston near Edinburgh, and told him among other things of a new invention in Denmark (by Longomontanus as 'tis said) to save the tedious multiplication and division in astronomical calculations. Neper being sollicitous to know farther of him concerning this matter, he could give no other account of it than that it was by proportional numbers. Which hint Neper taking, he desired him at his return to call again upon him. Craig after some weeks had passed did so, and

" Neper.

“ Neper then shewed him a rude draught of what he called *Canon Mirabilis Logarithmorum*: which draught with some alterations he printing in 1614, it came forthwith into the hands of our author Briggs, and into those of Will. Oughtred, from whom the relation of this matter came.”

THIS story is either entirely a fiction, or much misrepresented. There is no mention of it in Oughtred's writings *. There are no traces of the logarithms in the works of Longomontanus †, who was a vain man and survived Napier twenty nine years ‡, without ever claiming any right to the invention of those numbers, which had for many years been universally used over Europe.

THE following hypothesis may perhaps obviate the story of Anthony Wood. Might not Craig, whom reason and Tycho Brahe could not divest of the prejudices of the Aristotelian philosophy which he had imbibed, on returning to Edinburgh from Denmark, visit Napier and tell him among other literary news that Longomontanus had invented a method of avoiding the tedious operations of multiplication and division in the solution of triangles? After getting the best answers this doctor could give to Napier's queries relative to this method, I perceive, says the baron of Merchiston, that Longomontanus hath invented, improved, or stolen from the *Fundamentum Astronomicum*, the Prosthaphæresis of Raymar: but if you will take the trouble of calling upon

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* Oughtred's *Clavis Math.* Oxon 1677, &c. † Smith, *Briggii vita*, and Ward's *lives. Art.* Briggs.

‡ Vossius (*de Nat. Artium*) cited by Ward, places the death of Longomontanus in the year 1647.

me some time hence, I will shew you a method of solving triangles by proportional numbers quite distinct from this we have been talking of; which method came into my head some short time ago, and will require many years intense thinking and labour to bring it to perfection. Accordingly a few weeks afterwards, when Craig returned to Merchiston, Napier shewed him the first rude draught of the *Canon Mirificus*. Craig, having occasion to write very soon after to Tycho Brahe, mentioned to him this work without saying any thing about its author *.

JUSTUS Byrgius also, instrument maker and astronomer to the Landgrave of Hesse, a man of real and extraordinary merit, is said by Kepler, in his *Tabulæ Rudolphæ*; to have made a discovery of the Logarithms, previous to the publication of the *Canon Mirificus*. The passage referred to is as follows: "Apices logistici, Justo Byrgio, multis annis ante editionem Neperianam, viam præiverunt ad hos ipfissimos logarithmos (i. e. Briggianos) et si homo cunctator et secretorum suorum custos, fætum in partu destituit, non ad usos publicos educavit. That is the accents (', ", "", "", &c. denoting minutes, seconds, thirds, fourths, &c. of a circular arch) led Byrgius to the very same logarithms (now in use) † many years before Napier's work appeared: but Justus being indolent and reserved (or jealous) with regard to his own inventions, forsook this his offspring (at or) in its birth, and trained it not up for public service.

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* Nihil autem (writes Kepler to Crugerus) supra Neperianam rationem esse puto: et si quidem, Scotus quidam, literis ad Tychonem anno 1594, scriptis, jam spem fecit Canonis illius Mirifici. Kep. Epist. a Gottheb. Hantsh. folio p. 460.

† Thus Byrgius might conceive Log. $a^0 = 0$

Log. $a^1 = 1$

Log. $a^2 = 2$

Log. $a^3 = 3$ &c a being any number less than 60.

It may be observed that this affair rests on the single testimony of Kepler; but it would perhaps be considered as a species of heresy to doubt the testimony of so great a man. It has been insinuated, however, that from partiality to a countryman he might imagine he saw more than was really to be found in the papers of Byrgius *. Indeed the expression, *fatum in partu defituit*, gives a colour of truth to the insinuation, and tempts one to think, that Justus' acquaintance with the logarithms, was much on a par with that of Stifellius. Moreover, it is highly probable, that even the profound and penetrating Kepler might have perused the manuscripts of Byrgius, without paying any particular attention to his principles of the logarithms, had he himself not been previously acquainted with Napier's theory of those numbers. Neither does it seem probable that Byrgius, had he known its value, could have been so indolent, so unreasonably reserved, and so dead to the sense of reputation, as to conceal from all the world an invention so useful and so glorious. We know also, that he communicated to his scholars and others a most ingenious and easy method of constructing the tables of the natural sines †. But setting all this entirely aside, and granting a great deal more in favour of Byrgius than Kepler's words impute to him; nothing can thereby be detracted from the merit of Napier, who never saw or heard of Byrgius' pretended discovery of the logarithms; for, by Kepler's own confession, *bomo cunctator et secretorum suorum custos, hoc inventum non ad usus publicos educavit.*

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* Montucla Histoire des Mathematiques.

† This method is unfolded, and dedicated to Justus Byrgius its inventor by Raymar in his *Fundamentum Astronomicum*. See also a part of a letter of Rothmannus to Tycho Brahe in Gaffendi Vit. Tych. anno 1592.

It is therefore upon clear and indubitable evidence that, *cum de aliis
sere omnibus praeclaris inventis plures contendant gentes, omnes Neperum loga-
rithmorum authorem agnoscunt qui tanti inventi gloria solus sine æmulo fruitur* * ; while several nations contend for almost every other famous invention, all agree in recognising Napier as the unrivalled author of the logarithms, and as solely entitled to the glory of so great a discovery.

SECTION

* Keil de Log. Praef.

getting the logarithms of the other primary numbers, and then by multiplying the logarithms of the other numbers by the powers of 2, 3, 4, &c. we get the logarithms of all the composite numbers.

It is to be observed that the logarithms of the primary numbers are to be found by the method of the first section.

SECTION IV.

NAPIER'S METHOD OF CONSTRUCTING THE LOGARITHMIC CANON.

HAD Napier's principal idea been to extend his logarithms to all arithmetical operations whatever, he would have adapted them to the series of natural numbers, 1, 2, 3, 4, &c. In that case, having considered the velocity of the two points as continuing the same for a very small space of time, after setting out from N and L (Fig. XI.), he would have taken Nn itself as the logarithm of $CN + Nn$, or Cn . Now as Cn surpasses CN or unity by a very small quantity, it is evident that, when raised to its successive powers, there will be found in the several products such as are very near in value to the natural numbers 1, 2, 3, 4, &c. agreeably therefore to the above theory (Sect. III. prop. 9.) Nn being equal to d , and x being a positive integer, any natural number may be represented by $(1+d)^x$ and its logarithm in Napier's system by $x d$.

By this formula might the logarithms of all the primary numbers 3, 5, 7, &c. be calculated; from which those of all the composite numbers 4, 6, 8, 9, 10, &c. are easily deduced by simple additions (Sect. III. prop. 7.) or by multiplications by 2, 3, 4, 5, &c. (Sect. III. prop. 9).

NAPIER'S.

NAPIER's views were entirely confined to the facilitating of trigonometrical calculations. This is the reason of his calculating only the logarithms of the sines; the log. of any given number being easily deduced from these by means of a proportion.

IN order to effect his purpose, he considered that the radius, or sine total, being supposed to consist of an infinite number of infinitely small and equal parts, all the other sines would be found in the terms of a geometrical series descending from it to infinity; and that the logarithm of the radius being supposed equal to zero, the logarithms of all the series, beginning with the radius, would be found in the terms of an arithmetical series, ascending from zero to infinity by steps equal to the logarithm of the ratio in which the geometrical series descends.

AGREEABLY to this idea, he supposes the radius = CN = 10000000, and first constructs three tables, of which the first contains a geometrical series descending from the radius to the hundredth term in the ratio of 10000000 to 9999999. It is formed by a continual subtracting, from the radius and every remainder, its 10000000th part. The decimals in every term are pushed to the seventh place: a specimen of this table follows.

10000000

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10000000 . 0000000
1 . 0000000
9999999 . 0000000
9999999
9999998 . 0000001
9999998
9999997 . 0000003
9999997
9999996 . 0000006
and so on to
9999900 . 0004950

THE second table contains a geometrical series descending from the radius to the fiftieth term, in the ratio of 100000 to 99999, nearly equal to that of the first term 10000000 . 0000000 to the last 9999900 . 0004950 of the first table. It is formed by a continual subtracting, from the radius and every remainder, its 100000th part. The decimals are pushed to the sixth place. A specimen of this table follows.

10000000 . 0000000
100 . 000000
99999900 . 000000
99 . 999000
9999800 . 001000
99 . 998000
9999700 . 003000
99 . 997000
9999600 . 006000
and so on to
9995001 . 222927

THE third table consists of sixty nine columns of twenty one terms each. The first column contains a geometrical series descending from the radius in the ratio 010000 to 9995, nearly equal to that of the first term 10000000 . 000000 to the last 999501 . 222927 of the second table. It is formed by a continual subtracting, from the radius and every remainder, its 2000th part. The decimals are pushed to the fourth place. The first, second, third, &c. terms of all the other columns are geometrical serieses descending from the first, second, third, &c. terms of the first column to the first, second, third, &c. terms of the last column, in the ratio of 100 to 99, nearly equal to that of the first term 10000000 . 0000 to the last 9900473 . 57808 of the first column. Each of these serieses is formed by a continual subtracting, from the first, second, third, &c. term of the first column and from every remainder, its 100th part. The decimals are pushed to the fourth place. A specimen of this table follows.

10000000 . 00000	
5000 . 0000	
<hr/>	
9995000 . 00000	
4997 . 50000	
<hr/>	
9990002 . 50000	
4995 . 00125	
<hr/>	
99985007 . 49875	
4992 . 50374	
<hr/>	
9980014 . 99501	
and so on to	
9900473 . 57808	

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HAVING generated the above proportionals in this ingenious and easy way, he finds their logarithms in the following manner:

By prop. 14. or 15. where $CN=10000000$ and $Cn=9999999$ he finds Ll' (the log. of 9999999) = 1.0000005: the logarithm therefore, by prop. 9. of (the last or 100th term of the first table) 9999900. 0004950 is 100. 000005.

By prop. 16. or 17. where $Cn'=99999900. 0004950$ $Cn''=9999900. 0000000$ and $Ll'=100. 000005$, he finds $l1''$ (the difference of the logarithms of Cn' and Cn'') = 0. 0004950; the logarithm therefore (of 99999) Ll'' is 100. 0005; and by prop. 9. the logarithm of (the last or 50th term of the second table) 999501. 222927 is 5000. 025.

By the same 16th or 17th proposition, he found the logarithm of 9995000. 0000 = 5001. 2485387; for, by the continual addition of this logarithm to itself (prop. 7), he obtained all the logarithms of the numbers of the first column of the third table. In the same manner he found the logarithms of all the first, second, third, &c. terms of the other columns of this table.

THUS did Napier complete, what he calls, the Radical Table, of which a specimen follows:

RADICAL.

LIFE, WRITINGS, AND
RADICAL TABLE.

FIRST COLUMN.

SECOND COLUMN.

NATURAL.	ARTIFICIAL.	NATURAL.	ARTIFICIAL.
10000000 . 0000	0	9900900 . 0000	100503 . 3
9995000 . 0000	5001 . 2	9895050 . 0000	105504 . 6
9990002 . 5000	10002 . 5	9890102 . 4750	110505 . 8
9985007 . 4987	15003 . 7	9885157 . 4237	115507 . 1
9980014 . 9950	20005 . 0	9880219 . 8451	120508 . 3
and so on to	and so on to	and so on to	and so on to
9900473 . 5780	100025 . 0	9801468 . 8433	200528 . 4

and so on to

COLUMN 69.

NATURAL.	ARTIFICIAL.
5048858 . 8900	6834225 . 8
5046333 . 4605	6839227 . 1
5043811 . 2932	6844228 . 3
5041289 . 3879	6849229 . 6
3038768 . 7435	6854230 . 8
and so on to	and so on to
499869 . 4034	6934250 . 8

THE numbers and logarithms in the above table, coinciding nearly with the natural and logarithmic sines of all the arcs from 90° to 30° , he was enabled, by means of prop. 16. or 17. and a table of the natural sines, to calculate the logarithmic sines to every minute of the last 60° of the quadrant.

IN order to obtain the logarithms of the sines of arcs below 30° , he proposes two methods.

THE first is this. He multiplies any given sine of an arc less than 30° by the number 2, 10, finding the logarithms of the numbers 2 and

and 10 by means of the radical table, or takes some one of the compounds of these so as to bring the product within the compass of the radical table. Then having found, in the manner before described, the logarithm of this product, he adds to it the logarithm of the multiple he had made use of; the sum is the logarithm sought.

THE second method is derived from a property of the sines which he demonstrates. The property is this: Half the radius is to the sine of half an arc, as the sine of the complement of half that arc is to the sine of the whole arc. Hence, as is evident from a foregoing prop. that the logarithm of the sine of half an arc may be had by subtracting the logarithm of the sine of the complement of half that arc from the sum of the logarithms of half the radius and of the sine of the whole arc.

By this second method, which is much easier than the first, the logarithms of the sines of the arcs below 45° may be obtained; those above 45° having been calculated by help of the radical table.

THE logarithms of the sines to every minute of the quadrant being found by the ingenious methods above described, the logarithms of the tangents were easily deduced by one simple subtraction of the logarithm of the sine of the complement from that of the sine for each arc. The logarithm of the radius, which so frequently occurs in trigonometrical solutions, having been very advantageously made equal to zero, the logarithms of all the tangents of arcs below 45° and of all the sines must have a different sign from that of the logarithms of

all the tangents of arcs above 45° . Napier chose the positive sign for the former which he calls *abundantes*, and the negative for the latter which he calls *defectivi*.

THE arrangement of the numbers in Napier's logarithmic table, is nearly the same with that neat one which is still in use. The natural and logarithmic sines and the logarithmic tangent of an arc and of its complement stand on the same line of the page. The degrees are continued forwards from 0° to 44° on the top, and backwards from 45° to 90° on the bottom of the pages. Each page contains seven columns; the minutes descend from $0'$ (to $30'$ or from $30'$) to $60'$ in the first, and from $60'$ (to $30'$ or from $30'$) to $0'$ in the last of these columns. The natural sines of the arcs, on the left and on the right hand, occupy the second and sixth column, and their logarithms the third and fifth respectively. The fourth column contains the logarithms of the tangents, which are taken positively if they refer to the arcs on the left, and negatively if they refer to the arcs on the right hand. A specimen of this table is here annexed.

INVENTIONS OF NAPIER.

61

Gr. 44 mi.	SINUS.	LOGARITHMI.	DIFFERENTIA.	LOGARITHMI.	SINUS.	
30	7009093	3553767	174541	3379226	7132504	30
31	7011167	3550808	168723	3382085	7130465	29
32	7013241	3547851	162905	3384946	7128225	28
33	7015314	3544895	157087	3387808	7126385	27
34	7017387	3540941	151269	3390672	7124344	26
35	7019459	3538989	145451	3393537	7122303	25
36	7021530	3536038	139633	3396406	7120261	24
37	7023601	3533089	133814	3399275	7118218	23
38	7025671	3530142	127996	3402146	7116175	22
39	7027741	3527197	122178	3405019	7114131	21
40	7029810	3524243	116359	3407894	7112086	20
41	7031879	3521311	110541	3410770	7110041	19
42	7033947	3518371	104723	3413648	7107995	18
43	7036014	3515432	98904	3415528	7105949	17
44	7038081	3512195	92886	3419409	7103902	16
45	7040147	3509560	87268	3422292	7101854	15
46	7042213	3506626	81450	3425176	7099806	14
47	7044278	3503694	75632	3428062	7097757	13
48	7046342	3500764	69824	3430940	7095708	12
49	7048406	3497835	64006	3433829	7093658	11
50	7050469	3494908	58178	3436730	7091607	10
51	7052532	3491983	52360	3439623	7089556	9
52	7054594	3489060	46543	3442517	7087504	8
53	7056655	3486139	40726	3445413	7085452	7
54	7058716	3483219	34908	3448311	7083399	6
55	7060776	3480301	29090	3451211	7081345	5
56	7062386	3477385	23273	3454112	7079291	4
57	7064695	3474470	17455	3457015	7077236	3
58	7066693	3475557	11637	3459920	7075181	2
59	7069011	3468645	5818	3462827	7071125	1
60	7071068	3465735	0	3465735	7071008	0

45
mi.
Gr.

IN the Appendix to the *Canonis mirifici constructio*, Napier delivers three other methods of computing the logarithms; but as these methods are generally better adapted to the construction of a species of logarithms different from that I have described, I shall postpone the account of them to the next section.

THE

THE ingenious method by which Napier constructed the radical table is almost peculiar to the species of logarithms it contains: It does not seem, however, to be susceptible of all the accuracy one would wish: for, notwithstanding the many precautions he had taken, particularly in pushing his numbers to several decimal places, the logarithms in his canon often differ from the truth by several units in the last figure. Of this he himself was apprised by finding different results from the two methods of determining the logarithmic sines of arcs under 30° . In order to remedy this defect, he proposes adding another zero to the radius; by which means, in pursuing this same method, the logarithms of the sines might be obtained true to an unit in the eighth figure.

SECTION

S E C T I O N V.

**THE COMMON LOGARITHMS DEVISED BY NAPIER AND PREPARED BY
BRIGGS, AND THE METHODS PROPOSED BY NAPIER FOR COMPUT-
ING THEM.**

ONE capital disadvantage attending the species of logarithms which first occurred to Napier, arises from the difference between the sign of the logarithms of the tangents of arcs greater than 45° and the sign of the logarithms of the sines of all the arcs of the quadrant.

THIS defect was easily remedied by supposing the smallest possible sine equal :o 1 and its logarithm o; as in this case, the logarithms of all the sines and tangents of every arc in the quadrant would have the same sign. But, if the same species of logarithms is made use of, the logarithm of the radius, which occurs so frequently in trigonometrical solutions, would be a number difficult to be remembered. More, therefore, would be lost than gained by this alteration. What species of logarithms will free us from a difference in the signs, and at the same time afford a logarithm of the radius that shall be easily remembered and easily managed? It was this very question, in all probability, that led to the common logarithms, which, of all others, are the best adapted

to our modern arithmetical notation. This system of logarithms has for its basis 1 as logarithm of the ratio of 10 to 1: so that the powers 1, 10, 100, 1000, &c. of the number 10 have their respective logarithms 0, 1, 2, 3, &c.* Here, by the bye, it may be observed, that not only Napier's manner of conceiving the generation of the logarithms, but his having computed that species of logarithms, which has been described, before the common logarithms occurred to him, *is a convincing proof of his not taking the hint of the logarithms from the remark of Stifellius*, formerly mentioned. I think it is even beyond doubt that Napier, in common with all other arithmeticians acquainted with the Arabic, or rather Indian figures, had observed that the product of any power of the number 10 by any other power of that number, was formed by joining or adding the zeros in the one to those in the other; and that the quotient of any one power of that number by any other, was formed by taking away or defacing a number of zeros in the dividend equal to the number of zeros in the division; and all this without thinking that he was, at that time, making the fundamental remark of the logarithms. Nor will this seem at all surprising to those who are acquainted with the history of science and of the human mind. It is seldom that we directly arrive at truth by the most natural and easy path.

Perhaps

* We have seen Sect. III. that in Napier's system the velocity of the two moveable points in N and L Fig. XI. is equal and that the logarithm $(Ll)^x$ of any number $(CN + Nn)^x$ or $1.0000000, 1^x$ is nearly equal to $(Nn)^x$ or $[.0000000, 1]^x$. In the common system the velocity at L is less than the half of the velocity at N; and the logarithm Ll of the number $[CN + Nn]^x$ [or $1.0000000, 1^x$] is nearly equal $[0.4342945] Nn x^x$ or $[0.0000000, 04342945]^x$: for in making this supposition the logarithm of 10, is found to be 1. The logarithms therefore in Napier's system are to the correspondent ones in the common system as 1 is to 0. 4342945 or, what is the same thing the common logarithms are to those of Napier as 1 is to 2.3025851.

Perhaps the strongest mark of the greatness of Napier's genius is not his inventing the logarithms, but his manner of inventing them. But to return; In this new system the radius was made equal to the 10th power 1000000000 of the number 10, of which the logarithm in the new scale is 10. The division of the radius into so great a number of parts, render the sine of the smallest sensible arc greater than 1, of which the logarithm is zero: consequently, the logarithms of all the sines and tangents of the arcs of the quadrant, being on the same side of zero, have the same sign.

WITH regard to the logarithm of the radius, its being easily managed is sufficiently obvious.

THUS in our common logarithms the disadvantages of Napier's system are avoided, whilst its advantages are retained and united to several others. Of these additional advantages in the common canon, the most capital is, that the units in the first figure (to which Briggs gave the name of characteristic) of the logarithm are fewer by one than the figures of the number to which that logarithm corresponds.

WHETHER Napier, or Briggs, *first* imagined this new species of logarithms, is a question which the learned do not seem as yet to have perfectly decided.

THE only evidence we have on which a decision can be grounded, is contained in the following particulars.

I. In a letter to Usher afterwards Archbishop of Armagh dated the 10th of March 1615, the year after the publication of Napier's Canon, Briggs writes thus *, "Napier lord of Merchiston hath set my head hand at work with his new and admirable logarithms: I hope to see him this summer if it please God; for I never saw a book which pleased me better, and made me more wonder."

II. In the dedication of his *Rabdologia*, published 1617, Napier has the following words, "Atque hoc mihi fini proposito, logarithmorum canonem a me longo tempore elaboratum superioribus annis edendum canavi, quicquid est in partibus numeris, et operationibus quae per eos fiunt, difficilioribus, alios substituit idem praestantes per faciles additiones, subtractiones, bipartitiones, et tripartitiones. Quorum siquidem logarithmorum speciem diam multo praestantiorem nunc inventimus, et creandi methodum, una cum eorum usu (si Deus longiorum vitæ et valetudinis usuram concesserit) evulgare statuimus; si ipsam autem novi canonis suppurationem, ob infirmam corporis nostri valetudinem, viris in hoc studii genere versatis relinquimus: impri- mis vero doctissimo viro D. Henrico Briggio Londini publico Geometriæ Professori, et amico mihi longe charissimo".

III. In the preface to the *logarithmorum abridg. prima*, a table of the common logarithms of the first thousand natural members, Briggs expresses himself in the following terms; "Why these logarithms differ from those set forth by their illustrious inventor, of ever respectful memory, in his *canon mirificus*, it is to be hoped, his posthumous work will shortly make appear."

IV.

* The life of Archbishop Usher and his correspondence, by Richard Par, D. D. 1686. folio, page. 36.

IV. In the preface the *Arithmetica Logarithmeca* *, there is the following paragraph, "Quod hi logarithmi diversi sunt (writes Briggs) ab iis quos clarissimus vir baro Merchistonii in suo edidit canone mirifico, non est quod mireri, enim meis auditoribus Londini publice in Collegio Greshamensi horum doctrinam explicarem; animadverti multo futurum commodius, si logarithmus sinus totius servatur o zero (ut in canone mirifico) logarithmis autem partii decimæ ejusdem sinus totius, nempe sinus 5 grad. 44 min. 21. secund. esset 1.00000,00000: atque ea de re scripsi statim ad ipsum. Authorem, et quamprimum hic anni tempus, et vacationem a publico docendi munere licuit, profectus sum Edinburgum; ubi humanissime ab eo acceptus hæsi per integrum mensem. Cum autem inter nos de horum mutatione sermo haberetur; *Ille se idem dudum sensisse*, et capuisse dicebat: veruntamen istos, quos jam paraverat, edendos curasse, donec alias, si per negotia et valetudinem liceret, magis commodos confecisset. If tam autem mutationem ita faciendam censem, ut o esset logarithmus unitatis et 1,00000 . 00000 sinus totius: quod ego longe commodissimum esse non potui non agnoscere". "Capi igitur ejus hortatu, rejectis illis quos antea paraveram, de horum calculo serio cogitare, et sequenti æstate iterum profectus Edinburgum, horum quos hic exhibeo præcipuos, illi ostendi. Idem etiam tertia æstate libentissime facturus, si Deus illum nobis tamdiu superstitem esse voluisset †."

IT may here be observed, that the manner in which Briggs proposed the application of the common logarithms to trigonometrical purposes,

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* Published in 1624.

† Ulacg in his title page to his edition of Brigg's log. writes to the same purport. "Hos numeros primis invenit clarissimus vir Joannes Neperus Baro Merchistonii; eos autem ex ejusdem sententia, manus, corumque ortum et usum illustravit Henricus Briggius".

did not at all tend to obviate the chief disadvantage of Napier's Canon: For according to Briggs' idea the sign of the logarithms of the sines and the tangents less than the radius must be the opposite of the sign of the logarithms of the tangents greater than the radius. It seems probable, therefore, that Briggs had been led to the common logarithms in endeavouring to get rid of the indirect method of finding the logarithms of the natural numbers by means of Napier's logarithmic Canon.

FROM the extracts above given it appears that the common logarithms had occurred to Napier before they occurred to Briggs: For the modesty and integrity of Napier's character put beyond dispute the truth of what he mentioned to Briggs at their first meeting, and to the Earl of Dunfermline in the dedication of the *Rabdologia*. But if the having first communicated an invention to the world be sufficient to entitle a man to the honour of having first invented it, Briggs has a better title than Napier to be called the inventor of this happy improvement of the logarithms *. For Briggs mentioned it to his pupils in Gresham College before the publication, in 1616, of Edward Wright's translation of the *Canon Mirificus*, in the Preface to which Napier gave the first notice of this improvement. With regard to the passage in the preface to the *Cbilias prima* published after Napier's death, where Briggs seems to require an acknowledgment from the editor of the *Canonis mirifici constructio*, that he had also imagined the new logarithms; the oversight or fault lies at the door of Napier's son and not at his own. Had Napier lived to publish his last mentioned work, it is hardly possible to entertain a shadow of doubt, but that he would have done ample

ple justice to Briggs in this particular. Napier and Briggs had a reciprocal esteem and affection for each other, and there is not the smallest evidence of there having existed, in the breast of either, the least particle of jealousy; a passion unbecoming and disgraceful in a man of merit.

We shall dismiss this affair with observing, 1. That after the invention of the logarithms, the discovery of the best species of logarithms was no difficult affair: 2. That the discovery of the common logarithms at that time, was a fortunate circumstance for the world, as there are few possessed of ingenuity and patience sufficient for the construction of such extensive and accurate tables as are those of Briggs' *Arithmetica Logarithmica*; and 3. That the invention of the new species of logarithms is far from being equal to some other of Briggs' inventions.

We come now to give a very brief description of those other methods of constructing the logarithms, proposed by Napier in the appendix to his posthumous work.

THE first of these methods is the following: The logarithm of 1 being supposed 0, and the logarithm of 10 1 followed by any number of zeros, 1000000000 for example; this last logarithm and the successive quotients divided (ten times) by the number 5 will give these (ten) logarithms 2000000000, 400000000, 80000000, 16000000, 3200000, 640000, 128000, 25600, 5120, 1024 to which the respective correspondent numbers may be found by extracting the 5th root, the 5th root of the 5th root, the 5th root of the 5th root of the 5th root, &c. of

the

the number 10. Then the last logarithm 1024 and the successive quotients divided (ten times) by the number 2, will give these (ten) logarithms 512, 256, 128, 64, 32, 16, 8, 4, 2, 1, &c. to which the respective correspondent numbers may be found, by extracting the square root, the square root of the square root, the square root of the square root of the square root, &c. of the number (found as above directed) corresponding to the logarithm 1024. By addition these (twenty) logarithms, and by multiplication their respective natural numbers serve for finding a great many other logarithms and their numbers.

THE second method is this: The logarithms (0 and 1000000000 for example,) of any two numbers 1 and 10 being given, the logarithm of any intermediate number (2 for example) may be found by taking continually geometrical means, first between one of these numbers (10) and this mean, then between the same number (10) and the last mean, and so on till there be found the number (2) wanted; of which the logarithm will be the corresponding arithmetical mean (3010299957) between the two given logarithms (0 and 1000000000).

THE third method is as follows: Suppose the common logarithm of a number not an integral power of 10 (2 for instance) find the number of figures in the 10th, 100th, 1000th, &c. power of that number: The successive numbers of figures (4, 31, 302, 3011, &c.) in these powers (2^{10} , 2^{100} , 2^{1000} , 2^{10000} , &c.) will always exceed by less than unity, but continually approach to the logarithm [30102999566, &c.] required.

THE

THE first of these methods is very operose, and by itself insufficient for constructing a complete logarithmic canon. The other two are much preferable. The last is particularly well adapted for finding the logarithms of the lower prime numbers: For, since the number of figures in the product of two numbers, is equal to the sum of the number of figures in each factor; except the product of the first figures in each factor be expressed by one figure only, which often happens; a few of the first, or left hand figures of the consecutive tenth powers of the given number, will suffice for finding the number of figures in these powers.

THIS last method depends on the distinguishing property of the common logarithms, which is, as was formerly observed, that the units in $[x]$ the rational logarithm of a number $[10^x]$ are one fewer than the number of figures in that number $[10^x]$. Whence it follows, that the units in the irrational logarithm of any other number are not quite one fewer than the number of integral figures in this other number. Now, as in a series of continued proportional numbers, the result of any two terms is the same, if one of the terms is raised to the power indicated by the exponent of the other, or if this other is raised to the power indicated by the exponent of the first; any number raised to the power indicated by the logarithm of 10 is equal to 10 raised to the power indicated by the logarithm of that number. If, therefore, [the logarithm of 10 being 10000, &c.] Y is any number not an integral power of 10 and y its logarithm, we shall have $Y^{10000, \&c.} = 10^y$ and the number of figures in $Y^{10000, \&c.}$ will exceed y by less than 1.

Слідчий зо змінами

S E C T I O N V I.

THE IMPROVEMENTS MADE ON THE LOGARITHMS.

THE improvements that have been made upon the logarithms after the death of their inventor, regard the theory, the methods of construction, the accuracy, extensiveness, and form of the tables of these numbers.

HOWEVER ingenious and beautiful Napier's manner of delivering the theory of the logarithms is, it must be acknowledged that it labours under one capital impropriety—treating geometrically a subject which properly belongs only to arithmetic. Sensible of this, Kepler *, Nicolas Mercator †, Halley ‡, Cotes ||, and other mathematicians of the first note, have treated the theory of the logarithms in a different and truly scientific manner. Their ideas are founded on the definition of the logarithms—*Numeri rationem exponentes*; which, although it is not expressly Napier's, is easily deducible from his theory. Thus, in a geometrical progression, having any finite number c greater than unity for it's basis, the exponent x is the logarithm of the ratio of the number c^x to c^0

or

* Chilias Logarithmorum 1624. Tab. Rudolph. 1627.

† Logarithmo technia, 1668.

‡ Phil. Trans. 1695.

|| Harmonia Mensurar. 1722.

or unity: And, if the quotient of two quantities is taken as the measure of their ratio, the definition is rendered more simple, and x will be the logarithm of c^x . Upon this principle is founded the analytical theory of the logarithms in the appendix.

IT was chiefly by the two last methods, described in the foregoing section, that Briggs constructed his logarithms. He invented also an original method of constructing logarithms by means of the first, second, third, &c. differences of given logarithms. How he came by it is not known. He describes it in his *arithmetica logarithmica* and there is a demonstration of it in Cotes's *Harmonia*, in Bertrand's *Mathematiques*, and in the works of a great many other authors.

EDMUND Gunter, Professor of Astronomy in Gresham College, who was the first that published a table of the logarithmic sines and tangents of that kind which Napier and Briggs had last agreed on, applied, in the year 1623, or 1624, the logarithms to a ruler which bears his name. This scale is of very great use in Navigation, and in all the practical parts of geometry where much accuracy is not required. On the account of this logarithmical invention, Gunter, after Napier and Briggs, has the best claim to the public gratitude.

AFTER Napier's death almost fifty years elapsed before the invention of the expressions of the logarithms by infinite serieses. Of these the three following, from which a great number of others are easily derived, were the first. *

Logarithm

* Appendix an. th. log.

$$\text{Logarithm of } (1+a) = a - \frac{1}{2}a^2 + \frac{1}{3}a^3 + \&c. \quad \text{--- X}$$

$$\text{Logarithm of } (1-a) = -a - \frac{1}{2}a^2 - \frac{1}{3}a^3 - \&c. \quad \text{--- Y}$$

$$\text{Logarithm of } \left(\frac{1+a}{1-a} \right) = a + \frac{1}{2}a^2 + \frac{1}{3}a^3 + \&c. \quad \text{--- Z}$$

THESE formulæ X, Y, and Z will converge the more quickly in proportion as a is supposed less than unity; and the sum of a few terms will generally suffice. They are the values of Napier's logarithms, but will represent every species of logarithms by being multiplied by an indeterminate quantity u , which is called the *modulus* of the system.

THE formula X was invented by Nicholas Mercator in the year 1667, and published in his *Logarithmotechnia* the year following. Gregory of St Vincent, about twenty years before, had shewn that one of the asymptotes of the hyperbola being divided in geometrical progression, its ordinates parallel to the other asymptote are drawn from the point of division, they will divide into equal portions the spaces contained between the asymptote and the curve: From this it was afterwards pointed out by Mersennus, that, by taking the continual sums of those parts there would be obtained areas in arithmetical progression corresponding to abscisses in geometrical progression, and consequently that these areas were analogous to a system of logarithms *. Wallis, after this, had remarked that the ordinate corresponding to the absciss a , counted on the asymptote of the equilateral hyperbola from a distance equal to the semi-axis 1, is equal to $\frac{1}{1+a}$; and he had demonstrated, in his *Aritmetica infinitorum* published in 1655, that the sum of $1^m + 2^m + 3^m + \&c. \quad \text{---} + a^m$ (a representing a finite quantity divided into an infinite number

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ber

* Hutton's Math. Tab.

ber of equal parts) is equal to $\frac{a^m + 1}{m + 1}$ *. With these data Mercator set himself to find the arca corresponding to the absciss a , or, what is the same thing, the logarithm of $(1 + a)$, which he happily accomplished by first developing, in the manner now commonly practised, the fraction $\frac{1}{1 + a}$ into $1 - a + a^2 - a^3 + \&c.$ which had not been attempted before: then, supposing a equal successively to 1, 2, 3, 4, &c. - - - - a , and lastly, taking successively the sums of all the zero, first, second, third, &c. powers of these numbers †.

IN the same year 1668 James Gregory, in his *Exercitationes Geometricæ*, gave a demonstration of Mercator's formula for the quadrature of the hyperbola different from his. He demonstrated the formula Y and found the formula Z by subtracting Y from X. He found too the value of $\log. (1 - a^2) = -a^2 - \frac{1}{2}a^4 - \frac{1}{4}a^6 - \&c.$ by adding Y to X: but this formula may be looked on as a solecism when applied to numbers: for the same result will be obtained by supposing a to be a square, in the formula Y, and even a more general result may be obtained by supposing a to be any power of a number.

SIR Isaac Newton, by his general method of the quadrature of curves, greatly simplified that of the quadrature of the asymptotic spaces of the equilateral hyperbola. The ordinate, (being as before =) $\frac{1}{1 + a}$ multiplied by a the fluxion of the absciss, becomes the fluxion of the corresponding asymptotic area: This product, developed in the manner of Mercator, is $a - aa + a^2a - a^3a + \&c.$ Taking the fluent of each term of.

* Montucla Hist. de Math.

† Appendix, Hyperbola.

of this series gives the fluent of the arca that is the logarithm of $1+x$ equal to X as before. It appears, from a letter of Newton's to Oldenburgh, that Newton had discovered the quadrature of the hyperbola by infinite but perhaps not general serieses, before the publication of the *Logarithmotechnia* *. Something of the same kind had also been discovered by Lord Brouncker †.

THE arcas of the equilateral hyperbola, as above described, exhibiting the logarithms of Napier's system, occasioned the appellation *hyperbolic* to his logarithms. It is difficult to account for the propriety of this epithet to Napier's logarithms; since not only the asymptotic arcas of equilateral but those of any other hyperbola may be made to represent every possible species of logarithms, by supposing, in the same hyperbola, the origin of the abscisses on the one asymptote at different distances from its intersection with the other. Thus the asymptotic arcas of the equilateral hyperbola will represent the common logarithms, if the origin of the abscisses is taken at the point on the asymptote where the ordinate is $u=0.43429$ &c. the distance of that point from the other asymptote being greater than the semi-axis but equal to $1\frac{1}{2}$. But if the origin of the abscisses is taken equidistant from the summit of the hyperbola and the intersection of the asymptotes, the asymptotes of the hyperbola, whose arcas represent the common logarithms, are inclined to each other about $25^{\circ} 44'$, of which the sine is $u=0.43429$ &c. ||

THE formulæ X and Y have also been deduced from the logarithmic §—a curve whose abscisses are the logarithms of its ordinates or

conversely

* Wallissi Opera. vol. 3. p. 634 and seq. cited by Hutton. † Montucla. ‡ Appendix, Hyperbola. || Hutton's Math. Tab. § Encyclopedie au mot Logarithmique. Appendix, Logarithmic.

conversely*. This curve is said to have been invented by Edmund Gunter†; but its properties, some of which are very remarkable, do not seem to have been much known and attended to till the time of Huygens, who enumerates them in his *Causa gravitatis*. It was considered afterwards by Leibnitz, Bernoulli, l'Hopital, and a great many others. The manner in which it is treated by John Keill in the tract on the logarithms subjoined to his edition of Euclid, facilitates very much the conception of these numbers. In the Appendix the reader will find this curve treated in a new manner, with an enumeration of some new properties.

THE same formulæ X and Y are easily deduced by the fluxionary method from Neper's generation of the logarithms. From what is said in a foregoing section it is evident that (Fig. XI.) the velocity of the first moveable at the point N is to its velocity at the point N' as CN is to CN'; but the velocity of the first moveable at the point N is the same with the velocity of the second moveable point at (any point of KΔ) L': therefore, in the language of fluxions, if $NN' = a$, $\text{Log. } (1+a) : a :: 1 : 1+a$, therefore $\text{Log. } (1+a) = \frac{a}{1+a} = a - aa + a^2 a - \&c.$ therefore $\text{Log. } 1+a = a - \frac{a^2}{2} + \frac{a^3}{3} - \&c.$ If the points n' and l' are taken, it may be shewn in the same manner that $\text{Log. } [1-a] = -a - \frac{a^2}{2} - \frac{a^3}{3} - \&c.$

IN the year 1695, Edmund Halley greatly improved the theory of the logarithms, by deriving the serieses for their construction from the principles

* Appendix, Logarithmic.

† Montucla.

principles of common Algebra independently of any curve. He was the first also, if I mistake not, that gave the general series for computing the numbers corresponding to given logarithms *. The analytical theory of logarithms, in the Appendix, is nearly on Halley's plan, but was materially finished before the author saw his treatise.

To describe, or enumerate, all the tables of logarithms, which have been published since the invention of these numbers, would be tedious and useless, and indeed next to impossible. We shall restrict ourselves to those which are the most considerable and the most useful.

IN the year 1624, Benjamin Ursinus, mathematician to the Elector of Brandenburg, published at Cologne, with his *Trigonometria*, a Table of Napier's logarithms of the sines to every ten seconds of the quadrant. He seems to have been at much pains in computing it, and, in order to obtain the logarithms true to the nearest unit in the eight figure, he supposed the radius followed by an additional zero, as Napier had advised †.

IN the same year, Kepler published, at Marpurg, his *Obelias Logarithmorum ad todidem numeros rotundos &c.* and, in the year following, a supplement to it. In this table, the logarithms are of the same kind with those of Napier, but adapted to *sines* in arithmetical progression.

SMALL tables of the same species of logarithms have been published by T. Simson in his fluxions, by Dr Hutton in his Math. Tab. and by a great many others, to eight places. In Euler's *Introductio in analysin*

X

infinitorum

* Phil. Trans. for 1695.

† Kepl. Epist.

infinitorum, there is a small table of the first ten natural numbers with their logarithms to twenty six places ; and, in Bertrand's work formerly mentioned, there are the logarithms of a great many of the first hundred natural numbers, and of several others, to the same number of places. Some of these differ from the truth, by some units only, in the last figure, and the logarithm of 61 is wrong in the sixteenth figure from the left hand. In the Appendix there is a table of Napier's logarithms of the first hundred and one natural numbers to twenty seven places.

IN the year 1624, Briggs published at London his *Arithmetica Logarithmica*. This work contains Briggs' or the common logarithms, and their differences, of all the natural numbers from 1 to 20000, and from 90000 to 100000 to fifteen places, including the index or characteristic. In some copies, of which there is one in the Library of the University of Edinburgh, there is added the logarithms of the numbers from 100000 to 101000, which Briggs had computed after the former had been printed off. Before his death, which happened in 1630, this author completed also a table of the logarithmic sines and tangents to fifteen places, for the hundredth part of every degree of the quadrant, and joined with it the natural sines, tangents, and secants, which he had before calculated. This work which Briggs had committed to the care of Henry Gellibrand, at that time professor of astronomy in Gresham College, was transmitted to Gouda, where it was printed under the inspection of Ulacq, and was published at Londen in 1633, with the title of *Trigonometria Britannica*.

THESE tables of Briggs' have not been equalled, for their extensiveness and accuracy together ; those of his logarithms that have been re-examined

examined having seldom been found to differ from the truth by more than a few units in the fifteenth figure.

IN the year 1628, Adrian Ulacq of Gouda, in Holland, after filling up the gap betwixt 20000 and 90000, which Briggs had left, republished the *Arithmetica Logarithmica*, together with a table of the logarithmic sines, tangents, and secants, to every minute of the quadrant. Some years afterwards, he published his *Trigonometria Artificialis*, containing Briggs' logarithms of the first twenty thousand natural numbers, and the logarithmic sines and tangents, with their differences for every ten seconds of the quadrant. In both these works, the logarithms are carried to the eleventh place including the index, and are held in much estimation for their correctness.

ABRAHAM Sharp, of Yorkshire, published with his *Geometry Improved*, in 1717, a table containing Briggs' logarithms of the first hundred natural numbers, and of all the prime numbers from 100, to 1100 and of all the numbers from 999980 to 1000020, to sixty two places including the characteristic. There is the greatest probability of all these logarithms being correct. The last forty-one [from 999980 to 1000020] were verified afterwards by Gardiner.

TABLES of the logarithms, carried to so great a number of places as those of Sharp, Briggs, and Ulacq, are seldom used; the logarithms to eight places inclusive of the characteristic being sufficient for all common purposes. The most useful tables are those which have the logarithms correct to the nearest unit in the eight figure, disposed so as to

take

take up little room, and, at the same time, to afford the easiest and most speedy means of finding the intermediate logarithms, or numbers corresponding to given numbers or logarithms. The form of the tables best adapted to answer these purposes was first introduced by Nathaniel Roe, a clergyman in Suffolk, in his *Tabulæ Logarithmicae*, printed at London in 1633. This form was improved by John Newton, in his *Trigonometria Britannica* published at London in 1658, and by Sherwin in his Mathematical Tables, of which the first edition was printed in 1705. It has received additional improvements in Mr Callet's edition of Gardiner's Tables printed at Paris in 1783.*

THE dispositon of the tables is as follows: Each page of the logarithms of the natural numbers is divided into twelve columns. The first column, titled N at top and bottom, contains the natural number. In the second column, marked O, are the logarithms, without the characteristic, of these numbers: the three first figures, belonging to the logarithms of more numbers than one, are separated by a point from the other four figures of the logarithm of the *first* of these numbers and are left out before the other four figures of the logarithms of the rest. In each line of the next nine columns, marked with the nine significant digits 1, 2, 3, &c. are four figures, which, united to the first three isolated figures of the second column in the same line with them, or above them,

* Tables portatives de Logarithmes, publiques à Londres par Gardiner, Augmentées et perfectionnées dans leur disposition par M. Callet, et corrigées avec la plus scrupuleuse exactitude: contenant les logarithmes des nombres depuis 1 jusqu'à 102960, les logarithmes des sinus and tangentes, de seconde en seconde pour les deux premiers degrés et de 10 en 10 seconde pour tous les degrés du quart de cercle; précédées d'un précis élémentaire sur l'explication et l'usage des logarithmes et sur leur application aux calculs d'intérêts, à la Géométrie-pratique, à l'Astronomie et à la Navigation; suivies de plusieurs tables intéressantes et d'un discours qui en facilite l'usage. A Paris 1783.

them, form the logarithm of the number arising from the junction of the digit at top or bottom to the figures in the first column, corresponding to said four figures. When the last of the three first figures of a logarithm, corresponding to a number formed by figures in the first column and a significant digit at top, is found augmented by unity, these three figures, together with the correspondent fours, are moved a line downwards; by this means one avoids the mistaking one three figures for another, which, without special care, must often be the case in using Sherwin's, Gardiner's or Hutton's Tables. The last column contains the differences of the consecutive logarithms, together with the proportional parts corresponding to the nine digits. With these proportional parts one can compute by the eye alone the logarithms, not in the table, of all the numbers less than 1029600, and, with very little trouble more, those of all numbers less than 10296000, as exactly as eight places of figures can exhibit them. In the table of the logarithmic sines and tangents, the degrees and minutes are disposed nearly in the same manner as in Napier's Table. Each page contains eleven columns. In the first and last are the minutes. In the second and last but one are the seconds 0, 10, 20, 30, 40, 50, 0, and 0, 50, 40, 30, 20, 10, 0, of which the first and last zeros are in the same line with and the rest between each succeeding minute. In the third, fifth, seventh and ninth columns are the logarithmic sines or cosines, cosines or sines, tangents or cotangents, and cotangents or tangents, according as they refer to the degrees at top and the minutes and seconds in the first and second column, or to the degrees at the bottom and the minutes and seconds in the last penult columns. The other three columns contain the differences of these logarithms. The above description will become perfectly intelligible by inspecting the following specimens of these Tables.

TAB. DES LOG. DES NOMB. NAT.

N. 14800 L. 170

N	o	1	2	3	4	5	6	7	8	9	diff. part
N	o	1	2	3	4	5	6	7	8	9	
1408	170.2617	2911	3204	3497	3791	4084	4377	4670	4964	5257	5147
81	5551	5844	6137	6430	6723	7017	7310	7603	7896	8189	6176
82	8482	8775	9068	9361	9654	9947					7206
	171.						0240	0533	0826	1119	8235
83	1412	1704	1997	2290	2583	2876	3168	3461	3754	4046	1
84	4339	4632	4924	5217	5509	5802	6095	6387	6680	6972	3
1485	7265	7557	7849	8142	8434	8727	9019	9311	9604	9896	514
86	172.0188	0480	0773	1065	1357	1649	1941	2233	2526	2818	61
87	3110	3402	3694	3986	4278	4570	4862	5154	5446	5737	119
88	6029	6321	6613	6905	7197	7488	7780	8072	8364	8655	258
89	8947	9239	9530	9822							388
	173.				0113	0405	0697	0988	1280	1571	4117
1490	1863	2154	2446	2737	3028	3320	3611	3903	4194	4485	5146
.	
.	
.	
.	
N	o	1	2	3	4	5	6	7	8	9	

TAB. DES LOG. DES SIN. ET TANG.

12 DEG.

I	"	fin.	diff.	co-fin.	diff.	tang.	diff.	co-tang.	"	I
20	0	9.32959.88		9.98985.97		9.33973.91		0.66026.09	0	48
	10	9.32969.50	962	9.98985.51	46	9.33984.00	1009	0.66016.00	50	
	20	9.32979.13	963	9.98985.04	47	9.33994.09	1009	0.66005.91	40	
	30	9.32988.75	962	9.98984.58	46	9.34004.17	1008	0.65995.83	30	
			963		46		1008			
	40	9.32998.38	962	9.98984.12	46	9.34014.25	1008	0.65985.75	20	
	50	9.33008.00	961	9.98983.66	46	9.34024.33	1008	0.65975.67	10	
21	0	9.33017.61	961	9.98983.20	46	9.34034.41	1008	0.65965.59	0	39
			962		46		1008			
	10	9.33027.23	961	9.98982.74	46	9.34044.49	1007	0.65955.51	50	
	20	9.33036.84	961	9.98982.28	46	9.34054.56	1007	0.65945.44	40	
	•	•	•	•	•	•	•	•	•	•
	•	•	•	•	•	•	•	•	•	•
	•	•	•	•	•	•	•	•	•	•
	•	•	•	•	•	•	•	•	•	•
I	"	co-fin.	diff.	fin.	diff.	co-tang.	diff.	tang.	"	I

77 DEG.

THESE

THESE Tables, which are executed with a new and elegant type on good paper, form a small octavo volume. There is every probability in favors of their correctness. They are copied from the London edition of Gardiner printed in 1742, which is in the highest estimation for that quality. Messieurs Callet, Leveque and Prud'homme, three good mathematicians, revised the proof sheets, as did also the editor M. Jombert three several times. M. Didot senr. the printer formed the models of the types and founded them on purpose, and the editor avers that, during the course of the impression, none of the figures came out of their place; a precious advantage which he imputes to the justness of the principles that M. Didot has established in his foundery.

THERE is an additional improvement, which I am surprised none of the editors of our common logarithms has thought of making. What I allude to is the uniting, to the tables of the logarithms of the natural numbers and of the sines and cosines, the logarithms of their reciprocals (their arithmetical complements*, as they are called). By this means, all the common operations by logarithms might be performed by addition only, without any trouble. The logarithms of the natural numbers might be disposed on the left hand, and those of their reciprocals on the right hand pages. The characteristics of the latter, being equal to the difference between 10 and the number of integral figures in the natural numbers, would be as easily found as those of the former. The logarithms of the reciprocals of the sines and cosines might, in each page, be put in the same line with the logarithms of the sines and cosines,

having

* The arithmetical complements of the logarithms were first thought of by John Speedell, who, in his *new logarithms* first published in 1619, and several times afterwards, avoided the inconvenience of the signs in Napier's logarithms by that contrivance.

having their common differences between them, as the logarithms of the tangents and cotangents, which are reciprocals of each other, have theirs. It is very likely that the present edition of the *Tables portatives* will soon be exhausted. If, in a second edition, M. Jombert adopts the proposed amelioration, he will do an essential service to the community. 1. The computation might be accomplished, by a good arithmetician, in little more than three hours labour every day for half a year. 2. The type and length of the page being the same, the book would be little more than a fourth part thicker, and would still be of a convenient size.

IN the month of May, 1784, there were published proposals for publishing, by subscription, *A Table of Logarithmic sines and tangents, taken at sight to every second of the quadrant, accurately computed to seven places of figures besides the index: to which will be prefixed a table of the logarithms of numbers from 1 to 100000, inscribed, by permission, to the right honourable and honourable the Commissioners of the Board of Longitude, by Michael Taylor, one of the computers of the Nautical Ephemeris, and author of a Sexagesimal Table, published by order of the Commissioners of the Board of Longitude.* The plan of this work was submitted to the Board of Longitude, who came to a resolution to give Mr Taylor a gratuity of three hundred pounds sterling towards defraying the expence of printing and publishing it. This circumstance ought to be a sufficient recommendation of Mr Taylor, and it is to be hoped, that his laborious and useful undertaking will meet with the encouragement and recompence from the public which it so justly deserves. In the specimen annexed to the proposals, the degrees being as usual at the top and bottom of the page, the seconds occupy the

the first column: the minutes are disposed along the tops and bottoms of the other columns: immediately below the minutes at top stand the characteristics, and below *them* the three next common figures of the logarithms; the other four figures filling the columns. It is to be regretted, that an improvement, similar to M. Callet's, has not been adopted in this work, the printing of which was begun before the date of the proposals.

THE tables of logarithms which, with those that have been mentioned, are most in estimation, are those of the edition of Sherwin, which was corrected and published by Gardiner in the same year (1742) with his own tables—Those by Deparcieux*, and those of the small editions of Ulacq published at Lyons in 1670, and 1760 †.

THE London edition of Gardiner, which has been deservedly esteemed as containing the most accurate set of tables, is not entirely free from errors. There is, at the end of Dr Hutton's tables, a list of about fifty errors in the logarithms of the natural numbers, sines and tangents; twenty of which he himself discovered in collating the proofs of his book with the like parts of Gardiner's; all of these, however, that gentleman observes, are not in all the copies of this edition. In the Avignon edition of Gardiner (1770), the errors pointed out by Dr Hutton are above seventy. All the errors of the London edition are corrected in the *Tables portatives*, excepting that of the logarithm of the natural numbers 64445.

Z.

BEFORE

* Montucla.

† Hutton.

BEFORE concluding this section, we shall say a few words of the logarithms called logistic. The logistic logarithm of a number of seconds is the excess of the logarithm of 3600" above the logarithm of that number of seconds. A table of these logarithms was first given by Strut in his *Astronomia Carolina* published in 1661*. A similar one is given in several of the common logarithmic tables.

SECTION

¶ Tab. portativa.

S E C T I O N VII.

THE USE OF THE LOGARITHMS.

THE general use of the logarithms, as was before observed, is to convert every species of multiplication and division into addition and subtraction, and to raise quantities to any given power, and to extract their roots by easy multiplications and divisions. Examples of these operations, particularly in trigonometry, are prefixed to almost all the most considerable tables of logarithms. We beg leave to refer the reader to Gardiner, Callet, Sherwin, and Hutton, where he will find the theory, construction, and application of these numbers.

THE theory of the logarithms has put it in our power to solve, with great ease, an equation in algebra, which before could not be solved but with difficulty and tatonnement. In the equation $a^x = b$, if b is the unknown quantity, its value is found by multiplying a by itself as often as there are units in $x-1$: Again, if a is the unknown quantity, its value may be found by extracting the x th root of b . But if x is the unknown quantity, algebra, without the logarithms, can furnish no direct rule for finding its value. This, however, is easily accomplished by

by the assistance of the logarithms. Let L denote the logarithm of the quantity to which it is prefixed. Now since $a^x = b$, it is evident that

$$La = xLb: \text{ but } La^x = xLa: \text{ therefore } xLa = Lb: \text{ therefore } x = \frac{Lb}{La}.$$

THE Solution of Equations of the form $a^x = b$ is of great importance in political arithmetic. Suppose that a quantity at first m , being increased at the end of every equal portion of time by a quantity c , augments at the rate r ; and that it is found, at the end of a number x of these portions of time, to be augmented to n ; the equation expressing the relation of these quantities to each other is $(1+r)^x = n + \frac{c}{r}$.

$$m + \frac{c}{r}$$

By the help of the logarithms, this formula, among other purposes, serves for finding with facility in what time a sum of money n might be paid off by sinking at first a sum m , and at the end of every year another sum c , leaving their interest r to accumulate. In what time, for example, might the national debt of Great Britain, 270 millions of pounds Sterling, be extinguished by sinking one million every year and allowing its interest, five per cent per annum, to accumulate? The calculation is as follows.

$$n = 270 \text{ 000 000} \quad n + \frac{c}{r} = 290 \text{ 000 000} \quad \text{Log.} = 8.4623980$$

$$m = 1 \text{ 000 000} \quad m + \frac{c}{r} = 21 \text{ 000 000} \quad \text{Log.} = 7.3222193$$

$$\text{and } c = 1 \text{ 000 000} \quad \text{Log. } \left(\frac{n + \frac{c}{r}}{m + \frac{c}{r}} \right) = 1.1401787$$

$$r =$$

$$r = \frac{5}{100} = \frac{1}{20}$$

$$1+r = 1 + \frac{1}{20} = \frac{21}{20} \text{ Log. } 21 = 1.3222193$$

$$\frac{c}{r} = 20,000,000 \text{ Log. } 20 = 1.3010300$$

$$0.0211893 = \frac{1}{(1+r)} \text{ Log. } (1+r) = 0.0211893$$

$$x = \frac{1.1401787}{0.0211893} = \frac{11401787}{211893}$$

$$\begin{array}{r} \text{Log. } 11401787 = 7.0569729 \\ \text{Log. } 211893 = 5.3261167 \end{array}$$

$$\text{Log. } x = 1.7308562, x = 53.809$$

IN less than fifty four years, therefore, the British nation might get quit of their debt, if they could raise annually a million Sterling, over and above the amount of the interest of that debt and the expences of government.

THE same equation under the form

$$n = (m + \frac{c}{r}) \times (1+r)^x - \frac{c}{r}$$

serves for computing the number n of inhabitants of a country which, having at first m inhabitants, has received every year for x years a number c of foreigners, and has increased annually at the rate r . For example, suppose the number of the inhabitants of the United States of North America to be at present three millions, that they receive ten thousand emigrants yearly, and that the population in that country increases at the rate of one to twenty per annum; What will be the num-

A a ber

ber of inhabitants of those States a hundred years hence? The calculation is as follows:

$$m = 3000000 \quad m + \frac{c}{r} = 3200000 \quad \text{Log.} = 6.5051500$$

$$c = 1000000 \quad \text{Log.} (1+r) = \text{Log.} (1+r)^{100} = 2.1189300$$

$$r = \frac{1}{20} \quad \text{Log.} (m + \frac{c}{r}) (1+r)^x = 8.6240800$$

$$\frac{c}{r} = 200000 \quad (m + \frac{c}{r}) (1+r)^x = 420800000$$

$$x = 100 \quad \frac{c}{r} = 200000$$

$$n = 420600000$$

HENCE it appears, that were the lands of the United States extensive enough, and were the same circumstances, favourable to population as at present, to continue for one hundred years, the number of their inhabitants would amount to more than four hundred and twenty millions, which is a good deal greater than twice the number of inhabitants computed to be in all Europe.

THE logarithms also, after the invention of fluxions, give rise to a species of calculus called the exponential. This calculus was invented by John Bernoulli and first published in the year 1697*. It is founded on these two principles: 1. The logarithm of the power of a quantity is equal to the product of its exponent by the logarithm of its root, or $\log a^x = x \log a$: 2. The fluxion of the logarithm of a quantity is proportional to the quotient of the fluxion of that quantity by that quantity

* De Serie, Infinita. Jacobi Bernoulli.

ity or $L_a = \frac{a}{a}$. The exponential calculus is necessary for the investigation of curves, the exponents of whose abscisses and ordinates, or their functions in the equations to these curves, are themselves variable quantities, $u, v, z, \&c.$ Exponential curves, such, for example, as have for

the value of their ordinates $x^u, x^v, x^z, \&c.$ are said to be of the first, second, third, &c. order. What are the subtangents, curvatures, arcas, &c. of curves of this nature? Let SMM' , (Fig. XII.) be any curve, its abscis $CSP' = x$ and ordinate $P'M' = y$ and let there be another curve $\sigma\mu\mu'$ having the same absciss with the former, and its ordinate $P'M'\mu = z = x^y$ for example, let $p'\pi'v'$ be an ordinate infinitely near to $p'\mu'$ and $\mu'\pi'$ perpendicular to it, and let $\tau'\mu'$ be a tangent at the point μ' : the similar triangles $\tau'p'\mu'$ and $\mu'\pi'v'$ give $\tau'p':p'\mu':\mu'\pi':\pi'v'$, therefore the

subtangent $\tau'p' = \frac{z\dot{x}}{z}$: but $z = x^y$ therefore $Lz = Lx^y = yLx$ therefore $\dot{L}z$

$= \dot{y}Lx$ that is $\frac{\dot{z}}{z} = \dot{y}Lx + \dot{y}Lx = \frac{y\dot{x}}{x} + \dot{y}Lx$, and therefore $\tau'p' =$

$\frac{\dot{x}x}{y\dot{x} + x\dot{y}Lx}$. Hence it is evident that the relation of x to y , that is,

the Equation to the curve SMM' being given, the fluxion of y may be expressed by some function of x , and its fluxion may be obtained; which

value of the fluxion of y being substituted in the fraction $\frac{\dot{x}x}{y\dot{x} + x\dot{y}Lx}$

and the fluxion of x expunged from its numerator and denominator, there will be obtained a finite expression of the subtangent $\tau'p'$ of the curve

curve $\sigma\mu\mu'$. For example, let the curve SMM' be the logarithmic: we have $y = Lx$ *: therefore $\dot{y} = \frac{\dot{x}}{x}$; therefore $\tau'p' = \frac{x}{2Lx}$. From the value of the subtangent and from the equation ($z = x^{Lx}$ to the curve $\sigma\mu\mu'$) a great many of its properties are easily deduced. The ordinate $S\sigma$ at the summit of the curve is equal to the absciss CS: for $y = Lx = 0$ and $z = x = CS$. The tangent at the point σ is parallel to the axis CSD: for $Lx = 0$ and $\tau'p' = \frac{x^0}{20} = \frac{1}{200}$. The ordinate cs is an asymptote to the curve $\mu\mu\mu$: for $x = 0$ and $Lx = -\infty$ and therefore $\tau'p' = \frac{0}{200} = 0$. The tangent passing through the point c meets the curve $\sigma\mu\mu'$ at the extremity of the ordinate $z = \sqrt{x}$: for $x = \tau'p' = \frac{x}{2Lx}$; therefore $Lx = \frac{1}{2}$. The tangents to the points M and μ , where $y = \frac{1}{\sqrt{2}}$ and $z = x^{\frac{1}{\sqrt{2}}}$, meet in the same point τ in the axis: For the subtangent of the logarithmic c is $= xLx = \tau p = \frac{x}{2Lx}$; therefore $L^2x = \frac{1}{2}$ and $Lx = \frac{1}{\sqrt{2}}$. The curve $\sigma\mu\mu'$ may be called the *Numerico-Logarithmic*: and if the equation were $(Lx)^x = z$ or $y^x = z$ there would be generated a curve which might be called the *Logarithmo-numeric*.

THE above small specimen may suffice for giving an idea of the use of the exponential calculus. The reader will have observed that we have made use of Napier's, or, as they have been called, the natural logarithms. It would have been an easy affair to have made use of any other

* See Appendix.

other logarithms. It may here be observed that the logarithmic itself, is an exponential curve of the first degree or order: for the absciss x is of the form c^x , c being a constant quantity greater than unity and having 1 for its logarithm.

THOSE, who wish to enter fully into this subject, may consult the Works of John Bernoulli, and the *Analyse des Infinimens petits* of the Marquis de l'Hopital with M. Varignon's *Eclaircissements*.

ANOTHER use of the logarithms is to solve the problems of sailing according to the true chart, independant of a table of meridional parts. It was first published, by Mr H. Bond, about the year 1645, that *the meridian line was analogous to a scale of logarithmic tangents of half the complements of the longitudes* *. Nicolas Mercator seems to have been the first to demonstrate this property of the meridional line. But he kept his demonstration secret. James Gregory first published a demonstration of it in his *Exercitationes Geometricæ*. Halley, afterwards, (about the year 1695) gave a much better one in the philosophical transactions. On this subject the reader may consult Robertson's Navigation, where he will find it treated in a plain manner and illustrated with examples.

THE logarithms also exhibit the asymptotic arcs of the hyperbola †.

THEY are likewise of great service for the summation of infinite series in the calculus of fluents. This is true particularly of Napier's

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logarithms.

* Phil. Trans. N^o 219. † See Sect. vi. and Appendix.

logarithms. The sum, for example, of about seven hundred millions of terms of the infinite series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} +$, &c. is equal to 0.69314 718, Napier's logarithm of the number 2.

SECTION

S E C T I O N VIII.

NAPIER'S IMPROVEMENTS IN THE THEORY OF TRIGONOMETRY.

WE observed before that the Arabs, setting aside the chords of the double arcs, which rendered Trigonometry very complicated among the ancients, made use of the halves of these chords to which they gave the name of the *Sinus*. To that ingenious people we owe also the three theorems which are the foundation of our modern spherical trigonometry. By these theorems all the cases of rectangular spherical triangles and all the cases of oblique spherical triangles may be resolved, excepting when the three sides, or the three angles only, are the data. It was Regiomontanus who first invented two theorems for the solution of these two cases: by which means the theory of trigonometry was perfected. One of these theorems which serves for finding an angle from the three sides is, *The rectangle under the sines of the two sides of any spherical triangle is to the square of the radius; as the difference of the versed sines of the base and the difference of the two sides is to the versed sine of the vertical angle.* The other theorem, of itself, is not sufficient for the purpose of finding a side from the three angles.

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THIS last case, however, may be resolved into the former by means of the supplemental triangle, so called because its sides are the supplements of the angles of the other. This invention is due to Bartholomus Pitiscus*, who flourished in the beginning of the seventeenth century.

THE improvements made by Napier on this subject are chiefly three. 1. The general rule for the solution of all the cases of rectangular spherical triangles, and of all the cases of oblique spherical triangles, excepting the two formerly mentioned. 2. A fundamental theorem by which the segments of the base, formed by a perpendicular drawn from the vertical angle, may be found, the three sides being given. This, with the foregoing and the property of the supplemental triangle, serves for the solution of all the cases of spherical triangles. 3. Two proportions for finding by one operation *both* the extremes, the three middle of five contiguous parts of a spherical triangle being given.

THESE theorems are announced by Napier in terms to the following import:

1. Of the circular parts of a rectangular or quadrantal spherical triangle. *The rectangle under the radius and the sine of the middle part is equal to the rectangle under the tangents of the adjacent parts and to the rectangle under the cosines of the opposite parts.* The right angle or quadrant side being neglected, the two sides and the complements of the other three natural parts are called the circular parts; as they follow each other as

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* Pitisco aliquid tribuo in *μετάτοιν* arcuum in angulos, et vicissim. Kep. Epist. 295.

it were in a circular order. Of these any one being fixed upon as the middle part, those next to it are the adjacent, and those farthest from it, the opposite parts.

2. *The rectangle under the tangents of half the sum and half the difference of the segments formed at the base by a perpendicular drawn to it from the vertical angle of any spherical triangle, is equal to the rectangle under the tangents of half the sum and half the difference of the two sides.*

3. *The sines of half the sum and half the difference of the angles at the base of any spherical triangle are proportional to the tangents of the half base and half the difference of the sides.*

4. *The cosines of half the sum and half the difference of the angles of the base of any spherical triangle, are proportional to the tangents of half the base and half the sum of the sides.*

NAPIER gives also the two following theorems for finding an angle, the three sides of any spherical triangle being given.

5. *The rectangle under the sines of the two sides is to the rectangle under the sines of half the sum and half the difference of the base and the difference of the two sides, as the square of the radius is to the square of the sine of half the vertical angle.*

6. *The rectangle under the sines of the two sides is to the rectangle under the sines of half the sum and half difference of the sum of the two sides and the base, as the square of the radius is to the square of the cosine of the vertical angle.*

FOR the demonstration of the various cases of the first of these six propositions, he refers to the elementary books on trigonometry then in use. This proposition is not so susceptible of a direct demonstration. The demonstration perhaps the nearest to a direct one is given in the appendix ; of which demonstration the hint is taken from Napier.

His demonstration of the second proposition is extremely elegant and of an uncommon cast. The reader on these accounts, it is presumed, will be very glad to see the substance of it ; which is as follows :

LET a plane MN (Fig. XIII.) touch the sphere ADP at the point A, the extremity of its diameter PA. Upon the surface of the sphere let there be described the triangle $A\lambda\gamma$ acute in γ , or $A\lambda\epsilon$ obtuse in ϵ . Let the sine $A\lambda$ and the base $A\gamma$ or $A\epsilon$ be produced to the point P. With the pole λ and distance $\lambda\gamma$ or its equal $\lambda\epsilon$ let the small circle of the sphere $\epsilon\gamma$, intersecting λP in ϵ and λA in γ , be described : and from λ let the arc $\lambda\mu$ be drawn perpendicular to $A\epsilon\gamma$. $A\gamma$ is the sum of the segments of the base and $A\epsilon$ their difference. $A\epsilon$ is the sum of the sides and $A\gamma$ their difference. Let there be supposed a luminous point in P: The shadows, A, b, and c, of the points A, ϵ and γ , upon the plane MN, are in the same straight line, because the points A, ϵ , γ and P are in the same circular plane : also the shadow A, d and e, of A, γ and ϵ , upon the plane MN, are in the same straight line, because A, γ , ϵ and P are in the same circular plane. Since PA is perpendicular to the plane MN, the plane triangles PAC, PAB, PAe and PAD are rectangular in A: therefore, to the radius PA, the straight lines Ac, Ab, Ae and Ad, are the tangents of the angles APC or AP γ , APB or AP ϵ ,

APE

APe or Ap ε and APd or AP δ respectively. But these angles, being at the circumference of the sphere, have for their measures the halves of the arcs intercepted by their sides: therefore Ac, Ab, Ae and Ad are the tangents of the halves of A γ , A ℓ , A ε and A δ respectively. Now (by optics) the shadow of any circle, described on the surface of the sphere, produced by rays from a luminous point situated in any point of that surface excepting the circumference of the circle, forms a circle on the plane perpendicular to the diameter at whose extremity the luminous point is placed: therefore the points c, b, e and d are in the circumference of a circle: therefore $Ac \times Ab = Ae \times Ad$. Q. E. D.

THE third and fourth propositions are not demonstrated by Napier. He probably deduced them from the second in a manner similar to that in the appendix; where the reader will find all of these and some other theorems of the same kind, demonstrated. Napier had left the third proposition under a clumsy form. It was put into the form above given by Briggs in his *Lucubrationes* annexed to the *Canonis Mirifici Constructio*. This circumstance is not the sole mark of this work being a posthumous publication.

THE fifth proposition is deduced by Napier from the theorem of Regiomontanus, and it is likely he derived the fifth from the same source. To these two theorems the logarithms are much more applicable than to that of Regiomontanus.

SINCE Napier's time the chief improvement made in the theory of trigonometry is the application of the calculus of fluxions to it; for which we are indebted to Cotes.

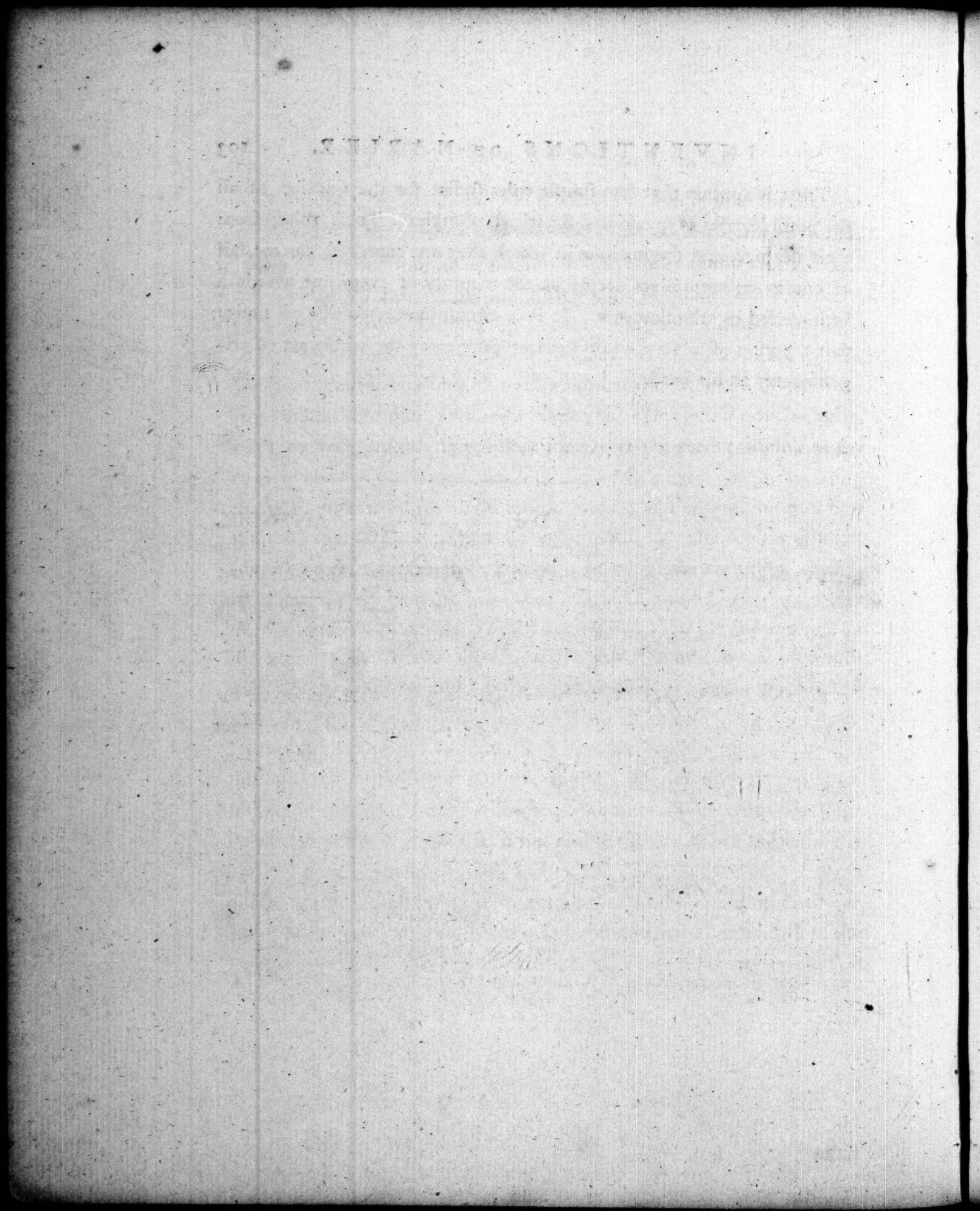
M. PINGRE, in the *Memoires de mathematique et de physique* for the year 1756, reduces the solution of all the cases of spherical triangles to four analogies. These four analogies are in fact, under another form, Napier's Rule of the circular parts and his second or fundamental theorem, with its application to the supplemental triangle. Although it would be no difficult matter to get by heart the four analogies of M. Pingre, yet there are few blessed with a memory capable of retaining them for any considerable time. For this reason, the rule for the circular parts, ought to be kept under its present form. If the reader attends to the circumstance of the second letters of the words *tangents* and *cosines* being the same with the first of the words *adjacent* and *opposite*, he will find it almost impossible to forget the rule. And the rule for the solution of the two cases of spherical triangles, for which the former of itself is insufficient, may be thus expressed: *Of the circular parts of an oblique spherical triangle, the rectangle under the tangents of half the sum and half the difference of the segments at the middle part (formed by a perpendicular drawn from an angle to the opposite side), is equal to the rectangle under the tangents of half the sum and half the difference of the opposite parts.* By the circular parts of an oblique spherical triangle are meant its three sides and the *supplements* of its three angles. Any of these six being assumed as a middle part, the opposite parts are those two of the same denomination with it, that is, if the middle part is one of the sides, the opposite parts are the other two, and, if the middle part is the supplement of one of the angles, the opposite parts are the supplement of the other two. Since every plane triangle may be considered as described on the surface of a sphere of an infinite radius, these two rules may be applied to plane triangles, provided the middle part be restricted to a *side*.

THUS

THUS it appears that two simple rules suffice for the solution of all the possible cases of plane and spherical triangles. These rules, from their neatness and the manner in which they are expressed, cannot fail of engraving themselves deeply on the memory of every one who is a little versed in trigonometry. It is a circumstance worthy of notice that a person of a very weak memory may carry the whole art of trigonometry in his head.

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APPENDIX.



A P P E N D I X.

I.

ANALYTICAL THEORY OF THE LOGARITHMS.

1. LET the consecutive terms of an infinite geometrical progression differ infinitely little one from another; it is evident that, any determined quantity c greater than unity being the basis of the progression, there will be some term $c^x=m$ any given quantity.

2. THE exponents of the terms of that progression are said to be the *logarithms* of those terms: Thus the symbol L denoting the logarithm of the quantity to which it is prefixed, $Lc^x=x$. Hence if $c^x=m$; then $Lm=x$ and $L\frac{1}{m}=-x=-Lm$.

THEOREM I.

3. The logarithm of a product is equal to the sum of the logarithms of its factors. For since $Lc^x=x$ and $Lc^z=z$ (2), it follows that $Lc^x+Lc^z=x+z$; but $x+z=Lc^{x+z}$ (2) $=Lc^x \times c^z$: therefore $Lc^x \times c^z=Lc^x+Lc^z$. Hence if $c^x=m$ and $c^z=n$ (1); then $Lmn=Lm+Ln$ and $L\frac{m}{n}=Lm-Ln$.

THEOREM.

THEOREM II.

4. *The logarithm of a power is equal to the product of its exponent by the logarithm of its root.* For, since $Lc^x=x$, it follows that $nLc^x=nx$; but $nx=Lc^{nx}(2)$, therefore $Lc^{nx}=nLc^x$. Hence if $c^x=m$, then $Lm^n=nLm$.

PROBLEM I.

5. *To exhibit the logarithm of a given number.* Since $c^0=1$, if d is an infinitely small quantity and μ any finite quantity, it is evident that $c^d=1+\frac{d}{\mu}$. Now $Lc^d=d(2)$, therefore $d=L(1+\frac{d}{\mu})$, therefore $id=iL(1+\frac{d}{\mu})=L(1+\frac{d}{\mu})^i$ (4). Let $(1+\frac{d}{\mu})^i=1+a$; we have $id=i\mu(1+a)^{\frac{1}{i}}-i\mu$; therefore, developing the surd quantity $(1+\frac{d}{\mu})^{\frac{1}{i}}$, making $i=\infty$, and reducing

$$L(1+a)=\mu(a-\frac{a^2}{2}+\frac{a^3}{3}-\&c) \quad \text{--- --- --- --- --- X}$$

Hence, if a is negative,

$$L(1-a)=-\mu(a+\frac{a^2}{2}+\frac{a^3}{3}+\&c) \quad \text{--- --- --- --- Y}$$

Hence, by subtracting Y from X

$$L(1+a)-L(1-a)=L(\frac{1+a}{1-a})=2\mu(a+\frac{a^3}{3}+\frac{a^5}{5}+\&c) \quad \text{--- Z}$$

6. The above formulæ are of no use for the calculation of the logarithms if a is supposed an integer. Let therefore m and n be any positive numbers, m being greater than n ; and

remo. Let $a=\frac{n}{m}$; then $1+a=\frac{m+n}{m}$, $1-a=\frac{m-n}{m}$ and $\frac{1+a}{1-a}=\frac{m+n}{m-n}$, and the formulæ X, Y, and Z become, by substitution, A, B, and C.

$$L(\frac{m+n}{m})=L(m+n)-Lm=\mu(\frac{n}{m}-\frac{n^2}{2m^2}+\frac{n^3}{3m^3}-\&c) \quad \text{--- A}$$

L

$$L\left(\frac{m-n}{m}\right) = L(m-n) - Lm = -\mu\left(\frac{n}{m} + \frac{n^2}{2m^2} + \frac{n^3}{3m^3} + \&c\right) - - - B$$

$$L\left(\frac{m+n}{m-n}\right) = L(m+n) - L(m-n) = 2\mu\left(\frac{n}{m} + \frac{n^3}{3m^3} + \frac{n^5}{5m^5} + \&c\right) - - - C$$

2do. Let $a = \frac{n}{m+n}$; then $1-a = \frac{m}{m+n}$, and the formula Y becomes D

$$L\left(\frac{m}{m+n}\right) = Lm - L(m+n) = -\mu\left(\frac{n}{m+n} - \frac{n^2}{2(m+n)^2} + \frac{n^3}{3(m+n)^3} + \&c\right) D$$

3to. Let $a = \frac{m}{m+n}$; then $1-a = \frac{n}{m+n}$, and the formula Y becomes E

$$L\left(\frac{n}{m+n}\right) = Ln - L(m+n) = -\mu\left(\frac{m}{m+n} + \frac{m^2}{2(m+n)^2} + \frac{m^3}{3(m+n)^3} + \&c\right) E$$

4to. Let $a = \frac{n}{2m+n}$; then $\frac{1+a}{1-a} = \frac{m+n}{m}$, and the formula Z becomes F

$$L\left(\frac{m+n}{m}\right) = L(m+n) - Lm = 2\mu\left(\frac{n}{2m+n} + \frac{n^3}{3(2m+n)^3} + \frac{n^5}{5(2m+n)^5} + \&c\right) F$$

5to. Let $a = \frac{n}{2m-n}$; then $\frac{1+a}{1-a} = \frac{m}{m-n}$, and the formula Z becomes G

$$L\left(\frac{n}{m-n}\right) = Lm - L(m-n) = 2\mu\left(\frac{n}{2m-n} + \frac{n^3}{3(2m-n)^3} + \frac{n^5}{5(2m-n)^5} + \&c\right) G$$

6to. Let $a = \frac{m-n}{m+n}$; then $\frac{1+a}{1-a} = \frac{m}{n}$ and the formula Z becomes H

$$L\left(\frac{m}{n}\right) = Lm - Ln = 2\mu\left(\left(\frac{m-n}{m+n}\right) + \frac{1}{2}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^5 + \&c\right) - - - H$$

7mo. Let $\frac{n^2}{m^2}$ be substituted for $\frac{n}{m}$ in the formula B: let this new formula be divided by c ; and Let $L(m^2-n^2)$ or $L(m+n)+L(m-n)=\sigma$ and $L(m+n)L(m-n)=\tau$: then shall

$$Lm = \frac{1}{2}\sigma + \frac{1}{2}\tau \left(\frac{n}{2m} + \frac{3}{12m^3} + \frac{5}{180m^5} + \frac{7}{7560m^7} + \frac{181n}{113400m^9} + \frac{11+173n}{113400m^9} + \&c \right) - - - I$$

REMARKS.

7. Of three quantities $m-n$, m and $m+n$, in arithmetical progression, the logarithm of the second, being given the logarithms of the other two may be found by one operation, if the odd and even powers of $\frac{n}{m}$ in the serieses A and B are calculated apart.

8. If n is supposed equal to unity, and if μ (the modulus of the system of logarithms to be afterwards determined), consists of a great number of figures, it will be much more convenient, in calculating by the serieses A, B, C, D, F, and G, to consider μ as the numerator of each term than as the multiplier of the sum of the terms.

9. The first step $\frac{2\mu n}{2m \times n}$ of the series F will give the logarithms of all numbers greater than 20000 true to fifteen places, if those of all numbers less than 20000 are given, and if $2\mu n$ does not exceed a few units.

10. The first step $\frac{r}{2} + \frac{s}{4m}$ of the series 1 will give the logarithms of all numbers greater than 10000 true to nineteen places, if those of all numbers less than 10000 are given, and if n does not exceed a few units.

THE reader will easily see that the logarithm of all numbers below m being known, that of $\frac{m+n}{2}$ and consequently that of $m+n$ and therefore σ as well as s will be known.

11. VARIOUS methods might be taken to compute with ease the logarithms of the lower prime numbers. The logarithms, for example, of about two thirds of the primes under 100 may be obtained with little trouble from a table of the continual halfs of the modulus, n being = 1. The inspection of the following table will make this evident.

given

given the logar. of.	sought	m	$m+1$	$m-1$	series
1	2	2		1	B
2	3	2	3	3	A
2 and 3	5	2^3	5	3	C
2 and 3	7	2^3	3^3	7	C
2,3 and 5	17	2^4	17	3×5	C
2 and 3	11	2^5	3×11		A
2	31	2^5		3^3	B
2,3,5 and 7	13	2^6	5×13	$3^3 \times 7$	C
2 and 3	43	2^7	3×43		A
2,3,5 and 7	19	$2^8 \times 10$	3×7	19	C
2,3,5 and 13	41	$2^8 \times 10$	41	3×13	C
2,3 and 5	79	$2^9 \times 10$	3^4	79	C
2,5 and 7	23	$2^9 \times 10$	7×23		A
2,3 and 5	53	$2^{10} \times 10$		3×53	B
2,5 and 11	20	$2^{10} \times 10$		11×29	B
2,3 and 5	71	$2^{11} \times 10$		$3^2 \times 71$	B
2,3,5 and 7	61	$2^{12} \times 10$	$3 \times 7 \times 61$		A

12. THE value of $L(1 + 2)$ was first given by Nicolas Mercator, who deduced it from a property of the equilateral hyperbola*. The series c was first demonstrated by James Gregory†. A series some what less general than i was produced by John Keill in his treatise *de Natura and arithmetic logarithmorum*: but I think I have some where seen it attributed to Newton. Some of the other formulæ I believe are new.

PROBLEM II.

13. To exhibit the modulus of a system of logarithms. This is effected by substituting c for m , and 1 for n , in the equation H. Its value is as follows :

I

$$\mu = \frac{1}{2\left(\frac{c-1}{c+1}\right) + \frac{1}{3}\left(\frac{c-1}{c+1}\right)^3 + \frac{1}{5}\left(\frac{c-1}{c+1}\right)^5 + \&c.}$$

REMARKS.

* Logarithmotechnia.

† Exer. Geom.

REMARKS.

14. IN our common system of logarithms, c is equal to 10; which gives the following values of μ and its reciprocal to thirty decimal places.

$$\mu = 0.43429\ 44819\ 03251\ 82765\ 11289\ 18917$$

$$\frac{1}{\mu} = 2.30258\ 50929\ 94045\ 68401\ 69914\ 54684$$

15. THE modulus of Napier's system is unity: for he supposed the logarithm of a number differing from unity by a very small quantity d to be equal to the sum or difference of 1 and d : Hence if 'L denote the common, or Brigg's, logarithm, and 'L, Napier's logarithm of the same number; then

$$'L = (0.43429 \&c) \cdot L; \text{ and } 'L = (2.30258 \&c) 'L$$

PROBLEM. III.

16. *To exhibit the number of a given logarithm.* We have seen that d being $= \frac{1}{\infty}$ and μ a finite quantity, that $c^d = 1 + \frac{d}{\mu}$, (5): we have

therefore $c^x = (1 + \frac{d}{\mu})^d$, and consequently

$$c^x = 1 + \frac{x}{\mu} + \frac{x^2}{1.2\mu^2} + \frac{x^3}{1.2.3\mu^3} + \&c \quad \Phi$$

and if x is negative,

$$c^{-x} = 1 - \frac{x}{\mu} + \frac{x^2}{1.2\mu^2} - \frac{x^3}{1.2.3\mu^3} + \&c \quad \Psi$$

Hence, by dividing Φ by Ψ ,

$$\frac{c_x}{c^{-x}} = c^{\frac{2x}{\mu}} = \frac{1 + \frac{x}{\mu} + \frac{x^2}{1.2\mu^2} + \frac{x^3}{1.2.3\mu^3} \times \&c}{1 - \frac{x}{\mu} + \frac{x^2}{1.2\mu^2} - \frac{x^3}{1.2.3\mu^3} \times \&c} \quad \Omega$$

17. If x is greater than μ the above serieses converge so slowly that that they are of no use for finding the number corresponding to a given

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given logarithm. Let therefore m and n be two numbers differing little from each other, m being greater than n , and

1 mo. Let $x = L\left(\frac{m}{n}\right) = Lm - Ln = s$. Then $c^s = \frac{m}{n}$ and $c^{-s} = \frac{n}{m}$ and the equations Φ and Ψ give

$$m = n\left(1 + \frac{\delta}{\mu} + \frac{\delta^2}{1.2\mu^2} + \frac{\delta^3}{1.2.3\mu^3} + \text{ &c}\right) \quad - \quad - \quad - \quad - \quad M$$

$$n = m\left(1 - \frac{\delta}{\mu} + \frac{\delta^2}{1.2\mu^2} - \frac{\delta^3}{1.2.3\mu^3} + \text{ &c}\right) \quad - \quad - \quad - \quad - \quad N$$

2do. Let $x = L\left(\frac{m}{n}\right)^{\frac{1}{2}} = \frac{1}{2}L\left(\frac{m}{n}\right) = \frac{1}{2}Lm - \frac{1}{2}Ln = \frac{1}{2}s$: then $c^{2s} = \frac{m}{n}$ and the equation Ω gives

$$m = n\left(\frac{1 + \frac{\delta}{2\mu} + \frac{\delta^2}{1.2.2^2\mu^2} + \frac{\delta^3}{1.2.3.2^3\mu^3} + \text{ &c}}{1 - \frac{\delta}{2\mu} + \frac{\delta^2}{1.2.2^2\mu^2} - \frac{\delta^3}{1.2.3.2^3\mu^3} + \text{ &c}}\right) \quad P$$

18. More generally, let there be any number n of numbers $m' < m'' < m''' < m'''' < \dots < m'''''$ which, taken consecutively, differ little from each other: and let $Lm^{[p]} - Lm^{[p-1]} = \delta^{[p-1]}$ and $\delta^{[p]}(n-2)\delta^{[p]} + \frac{(n-2)(n-3)}{1.2}\delta^{[p]} \dots$

$\pm \delta^{[p]} = \Delta^{[p]}$ (the quantities $[p]$, $[p-1]$, $[n]$, $n-1$ &c. inclosed in *lines*, expressing simply some terms of the series 1, 2, 3, 4, 5 &c.): we have

$$\frac{m''}{m'} - (n-2)\frac{m'''}{m''} + \frac{(n-2)(n-3)m''''}{1.2 \cdot m'''} \dots \pm \frac{m^{[n]}}{m^{[n-1]}} = \frac{\Delta'}{\mu} + \frac{\Delta''}{1.2\mu^2} + \frac{\Delta'''}{1.2.3\mu^3} + \text{ &c} \quad Q$$

REMARKS.

19. If the logarithms of the first 20000 natural numbers are given, the two first steps of the series $n\left(1 + \frac{\delta}{\mu} + \frac{\delta^2}{1.2\mu^2}\right)$ of the series M , or $n\left(1 - \frac{\delta}{\mu} + \frac{\delta^2}{1.2\mu^2}\right)$ of the series N , or the first step $n\left(\frac{2\mu+\delta}{2\mu-\delta}\right)$ of the series P , will give the number m or n true to about the fourteenth decimal place.

20. The serieses M and N were first given by Halley, in the Philosophical transactions for the year 1695. He exhibited also a series the same with P, but under an inclegant form; probably owing to his having deduced it from the actual division of M by N.

PROBLEM IV.

21. *To exhibit the number whose logarithm is equal to the modulus.* This is effected by the substitution of μ for x in the formula Φ . It's value is as follows

$$c'' = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \text{ &c}$$

or taking the sum of thirty fractional terms

$$c'' = 2.71828 \ 18284 \ 59045 \ 23536 \ 02874 \ 71353$$

NAPIER'S LOGARITHMS.

II.

A TABLE OF NAPIER'S LOGARITHMS,

OF ALL THE NATURAL NUMBERS FROM 1 TO 101 TO TWENTY
SEVEN PLACES.

Num.	LOGARITHMS.	Num.	LOGARITHMS.
1	0.00000.00000.00000.00000.00000.0	21	3.04452.24377.23422.99650.05979.8
2	0.69314.71805.59945.30941.72321.2	22	3.09104.24533.58315.85347.91757.0
3	1.09861.22886.68109.69139.52452.4	23	3.13549.42159.29149.69080.67528.3
4	1.38629.43611.19890.61883.44642.4	24	3.17805.38303.47945.61964.69416.0
5	1.60943.79124.34100.37460.07593.3	25	3.21887.58248.08200.74920.15186.7
6	1.79175.94692.28055.00081.24773.6	26	3.25809.65380.21482.04547.01795.6
7	1.94591.01490.55313.30510.53527.4	27	3.29583.68660.04329.07418.57357.1
8	2.07944.1546.79835.92825.16663.6	28	3.33220.45101.75203.92393.98169.8
9	2.19722.45773.36219.38279.04904.8	29	3.36729.58299.86474.02718.32720.3
10	2.30258.50929.94045.68401.79914.6	30	3.40119.73816.62155.37541.32366.9
11	2.39789.52727.98370.54406.19435.8	31	3.43398.72044.85146.24592.91643.3
12	2.48490.66497.88000.31022.97094.8	32	3.46573.59027.99726.54708.61666.1
13	2.56494.93574.61536.73605.34874.4	33	3.49650.75614.66480.23545.71888.2
14	2.61905.73296.15258.61452.25848.6	34	3.52636.05246.16161.38966.67667.4
15	2.70805.02011.02210.06599.60045.7	35	3.55534.80614.89413.67970.61120.8
16	2.77258.87222.39781.23766.89284.9	36	3.58351.89384.56110.00162.49547.2
17	2.83321.33440.56216.08024.95346.2	37	3.61091.79126.44224.44436.80956.7
18	2.89037.17578.96164.69220.77226.0	38	3.63758.61597.26385.76942.62595.5
19	2.94443.89791.66440.46000.90274.3	39	3.66356.16461.29646.42744.87326.8
20	2.99573.22735.53990.99343.52235.8	40	3.68887.94541.13936.30285.24557.0

APPENDIX.

Num.	LOGARITHMS.	Num.	LOGARITHMS.
41	3.71357.20667.04307.80386.67633.7	72	4.27666.61190.16055.31104.21868.4
42	3.73766.96182.83368.30591.78301.0	73	4.29045.94411.48391.12909.21088.6
43	3.76120.01156.93562.42347.28425.2	74	4.30406.50932.04169.75378.53278.0
44	3.78418.96339.18261.16289.64078.2	75	4.31748.81135.36310.44059.67639.1
45	3.80666.24897.70319.75739.12498.1	76	4.33073.33402.86331.07884.34916.8
46	3.82864.13964.89095.00022.39849.5	77	4.34380.54218.53683.84916.72963.2
47	3.85014.76017.10058.58682.09506.7	78	4.35670.88266.89591.73686.59648.0
48	3.87120.10109.07890.92906.41737.2	79	4.36944.78524.67021.49417.29455.4
49	3.89182.02981.10626.61021.07054.8	80	4.38202.66346.73881.61226.96878.2
50	3.91202.30054.28146.05861.87507.9	81	4.39444.91546.72438.76558.09809.5
51	3.93182.56327.24325.77164.47798.6	82	4.40671.92472.64253.11328.39955.0
52	3.95124.37185.81427.35488.79516.9	83	4.41884.06077.96597.92347.54722.3
53	3.97029.19135.52121.83414.44691.4	84	4.43081.67988.43313.61533.50622.2
54	3.98898.40465.64274.38360.29678.3	85	4.44265.12564.90316.45485.02939.5
55	4.00733.31852.32470.91866.27029.1	86	4.45434.72962.53507.73289.00746.4
56	4.02535.16907.35149.23335.70491.1	87	4.46590.81186.54583.71857.85172.7
57	4.04305.12678.34550.15140.42726.7	88	4.47733.68144.78206.47231.36399.4
58	4.06044.30105.46419.33660.05041.6	89	4.48863.63697.32139.83831.78155.4
59	4.07753.74439.05719.45061.60503.8	90	4.49980.96703.30265.06680.84819.3
60	4.09434.45622.22100.68483.04688.1	91	4.51085.95065.16850.04115.88401.9
61	4.11087.38641.73311.248-5.13891.1	92	4.52178.85770.49040.30964.12170.7
62	4.12713.43850.45091.55534.63964.5	93	4.53259.94931.53255.93732.44095.6
63	4.14313.47263.91532.68789.58432.2	94	4.54329.47822.70003.89623.81827.9
64	4.15888.30833.59671.85650.33927.3	95	4.55387.68916.00540.83460.97867.7
65	4.17438.72698.95637.11065.42467.8	96	4.56434.81914.67836.23848.14058.4
66	4.18965.47420.26425.54487.44209.4	97	4.57471.09785.03382.82211.67216.2
67	4.20469.26193.90966.05967.00720.0	98	4.58496.74786.70571.91962.79376.1
68	4.21950.77051.76106.69908.39988.6	99	4.59511.98501.34589.92685.24340.5
69	4.23410.65045.97259.38220.19980.7	100	4.60517.01859.88091.36803.59829.1
70	4.24849.52420.49358.98912.33442.0	101	4.61512.05168.41259.45088.41982.7
71	4.26267.98770.41315.42132.94545.3		

III.

TRIGONOMETRICAL THEOREMS.

(1) Lemma 1. The product of the radius by the difference of the versed sines of two arcs is equal to twice the product of the sines of half the sum and half the difference of those arcs.

$$R (\sin V, a - \sin V, b) = 2 \sin \frac{a+b}{2} \times \sin \frac{a-b}{2}.$$

(2) Corollary. The product of the radius by the versed sine of an arc is equal to twice the square of the sine of half that arc.

$$R \sin V, a = 2 \sin^2 \frac{1}{2}a.$$

(3) Lem. 2. The sum of the cosines of two arcs is to their difference as the cotangent of half the sum of those arcs is to the tangent of half their difference.

$$\text{Cof } a + \text{cof } b : \text{cof } a - \text{cof } b :: \cot \frac{a+b}{2} : \tan \frac{b-a}{2}.$$

(4) Lem. 3. The sum of the sines of two arcs is to their difference as the tangent of half the sum of those arcs is to the tangent of half their difference.

$$\sin a + \sin b : \sin a - \sin b :: \tan \frac{a+b}{2} : \tan \frac{b-a}{2}.$$

(5) Lem. 4. The sum of the cotangents of two arcs is to their difference as the sine of the sum of those arcs is to the sine of their difference

$$\cot a + \cot b : \cot a - \cot b :: \sin(b+a) : \sin(b-a).$$

A P P E N D I X.

(6) Lem. 5. The product of the sine of the sum of two arcs and the tangent of half that sum, is to the product of the sine of their difference and the tangent of half that difference, as the square of the sine of half their sum is to the square of the sine of half their difference.

$$\sin(a+b) \times \tan \frac{a+b}{2} : \sin(a-b) \times \tan \frac{a-b}{2} :: \sin^2 \frac{a+b}{2} : \sin^2 \frac{a-b}{2}.$$

(7) Lem. 6. The product of the sine of the sum of two arcs and the tangent of half their difference, is to the product of the sine of their difference and the tangent of half their sum, as the square of the cosine of half their sum is to the square of the cosine of half their difference.

$$\sin(a+b) \times \tan \frac{a-b}{2} : \sin(a-b) \times \tan \frac{a+b}{2} :: \cos^2 \frac{a+b}{2} : \cos^2 \frac{a-b}{2}.$$

(8) Lem. 7. In right angled spherical triangles the cosine of the hypothenuse is to the cotangent of one of the oblique angles as the cotangent of the other is to the radius.

(9) Lem. 8. In right angled spherical triangles the cosine of the hypothenuse is to the cosine of one of the sides as the cosine of the other is to the radius.

(10) Lem. 9. In any spherical triangle the product of the sines of the two sides is to the square of the radius as the difference of the versed sines of the base and the difference of the two sides is to the versed sine of the vertical angle, Fig. XIV.

$$\sin AB \times \sin BC : R^2 :: \sin V, AC - \sin V, (AB - BC) : \sin V, B *$$

(11) Lem. 10. In any spherical triangle the product of the sines of the two sides is to the square of the radius, as the difference of the versed sines of the sum of the two sides and the base is to the versed sine of the supplement of the vertical angle, Fig. XIV.

$$\sin AB \times \sin BC : R^2 :: \sin V, (AB + BC) - \sin V, AC : \sin V, \text{sup. B.}$$

(12)

* This is one of Regiomontanus' propositions.

(12) The natural parts of a triangle are its three sides and its three angles.

(13) The circular parts of a rectangular (or quadrantal) spherical triangle are the two natural parts adjoining to the right angle (or quadrant side) and the *complements* of the other three.

(14) Any one of these five being considered as a middle part, the two next to it are called the adjacent parts, and the other two the opposite parts: Thus, in the triangle dAB (Fig. XV.) rectangular in A, if the complement of the angle d is taken as a middle part, the adjacent parts are the side dA and the complement of the hypotenuse db ; and the opposite parts the side bA and the complement of the angle b .

(15) Of five great circles of the sphere AB, BC, CD, DE, and EA (Fig. XV.) let the first intersect the second; the second, the third; the third, the fourth; the fourth, the fifth; and the fifth, the first; at right angles in the points B, C, D, E and A: there are formed, by the intersections mentioned and by those at the respective poles a , b , c , d and e of these great circles, five rectangular triangles dAb , bDe , eBc , cEa and aCd : and, if these poles are joined by the quadrantal arcs ab , bc , cd , de and ea , there are formed five quadrantal triangles adb , dbe , bec , eca , and cad . The circular parts in all these triangles are the same: the *position* of these equal circular parts with respect to one another in each of these triangles is different: therefore

(16) What is true of the circular parts of a rectangular triangle is true of those of a quadrantal; and what is true of one middle part and its adjacent and opposite parts is true of the other four middle parts and their adjacent and opposite parts.

(17) The circular parts of an oblique spherical triangle are its three sides and the *supplements* of its three angles.

(18) Any one of these six being considered as a middle part, the two next to it may be called the adjacent parts; the one facing it, the remote part; and the other two, the opposite parts: Thus, in the triangle ABC (Fig. XIV.), if the side AC is taken as a middle part, the adjacent parts are the supplements of the angles A and C; the opposite parts, the sides AB and BC, and the remote part, the supplement of the angle B.

(19) Of six great circles of the sphere let the first three, AB, BC, and CA, intersect each other at the poles, B, C and A, of the second three, ca , ab and bc : the intersections, c , a and b , of the latter are the poles of the former: there are formed two triangles ABC and abc in which the circular parts are the same; the position of these equal circular parts is different in both: therefore

(20) What is true of one middle part and its adjacent, opposite, and remote parts, is true of any other middle part and its adjacent, opposite, and remote parts.

(21) If an arc $bBDd$ pass through the vertices of these two triangles, it will be perpendicular to their bases CDA and cda , and the segments at the base of the one triangle will be the complements of the segments at the vertical angle of the other: that is, $CD = 90^\circ - dba$, $AD = 90^\circ - abc$, $cd = 90^\circ - ABD$, $ad = 90^\circ - DBC$.

(22) If the radius of the sphere is supposed infinite, the sines and tangents of the sides of a triangle described on its surface, become the sides themselves of a plane triangle. Consequently all the formulæ of spherical trigonometry, where the sines and tangents only of the sides enter, are applicable to plane trigonometry. Those, however, in which any functions

functions of all the three angles and only one sine or tangent of one side enter, must be excepted.

(23) Of the circular parts we shall denote the middle one by M , the adjacent ones by A and a , and the opposite ones by O and o . If the triangle is oblique, the remote part we shall call m , and the segments at a side or angle (21) S and s .

(24) Theorem 1. Of the circular parts (13) of a rectangular (or quadrantal) spherical triangle, the product of the radius and the sine of the middle part, the product of the tangents of the adjacent parts and the product of the cosines of the opposite parts, are equal.

Demonstration. In the right angled spherical triangle dAb (Fig. XV.) we have $\text{cof } bd : \text{cot. } Abd :: \text{cot } Adb : R$ (8), and $\text{cof } bd : \text{cof } Ab :: \text{cof } Ad : R$ (9); therefore $R \times \text{cof } bd = \text{cot } Abd \times \text{cot } Adb = \text{cof } Ab \times \text{cof } Ad$; therefore (16)

$$R \times \sin M = \text{tang } A \times \text{tang } a = \text{cof } O \times \text{cof } o.$$

(25) Corollary 1. In any spherical triangle, the sines of the sides are proportional to the sines of the opposite angles. For, in the right angled triangles ADB and CDB (Fig. XIII.), $R \times \sin BD = \sin AB \times \sin A$, and $R \times \sin BD = \sin BC \times \sin C$; therefore $\sin AB : \sin BC :: \sin C : \sin A$.

(26) Cor. 2. In any spherical triangle, the sines of the segments of one of its sides (produced if necessary) are proportional to the cotangents of the angles at the extremities of that side. For, in the right angled triangles ADB and CDB , $R \times \sin AD = \text{cot } A \times \text{tang } BD$ and $R \times \sin DC = \text{cot } C \times \text{tang } BD$; therefore $\sin AD : \sin DC :: \text{cot } A : \text{cot } C$.

(27) Cor. 3. In any spherical triangle, the cosines of any two sides are proportional to the cosines of the segments of the third side. For, in the right angled triangles ADB and CDB , $R \times \text{cof } AB = \text{cof } AD \times \text{cof }$

H h

DB,

DB, and $R \times \cos BC = \cos CD \times \cos DB$; therefore $\cos AB : \cos BC :: \cos AD : \cos DC$

(28) Remark 1. This theorem serves for the solution of all possible cases of rectangular or quadrantal spherical triangles, and for the solution of all possible cases of oblique spherical triangles (by means of the arc drawn from one of its angles perpendicular on the opposite side); excepting when the three angles, or the three sides only, are the data.

(29) Rem. 2. This theorem, by confining the middle part to the two sides, (22) serves also for the solution of all possible cases of rectangular plane triangles, and for the solution of all possible cases of oblique angled plane triangles (by means of the perpendicular drawn from an angle to the opposite side); excepting when the three sides only are the data.

(30) Rem. 3. Were the complements of the two parts adjoining to the right angle or quadrant side and the other three natural parts taken as the circular parts, the theorem would be,

$$R \times \cos M = \cot A \times \cot a = \sin O \times \sin o.$$

But the other is preferable, because it is more easily remembered. The second letter of the word *tangent* is the same with the first of *adjacent*. It is the same of the words *cosine* and *opposite*. If this is attended to, it is hardly possible to forget the enunciation of the theorem.

(31) Theorem 2. Of the circular parts (17) of an oblique spherical triangle, the square of the sine of half the middle part, is to the square of the radius; as the product of the sines of half the sum and half the difference of the sum of the adjacent parts and the remote part, is to the product of the sines of the adjacent parts.

Dem. For since (Fig. XIV.) $\sin V. \text{supp. } B : R^2 :: \sin V, (AB + BC) - \sin V, AC : \sin AB \times \sin BC$ (11), it follows that $\sin \frac{V}{2} \text{ supp. } B : R^2 :: \sin$

$$\left(\frac{AB + BC + AC}{2} \right)$$

$$\left(\frac{\overline{AB}+\overline{BC}+\overline{AC}}{2}\right) \times \sin \left(\frac{\overline{AB}+\overline{BC}-\overline{AC}}{2}\right) \sin AB \times \sin BC (2 \text{ and } 1); \text{ therefore (20)}$$

$$\sin^2 \frac{1}{2} M : R^2 :: \sin \left(\frac{\overline{A}+\overline{a}+m}{2}\right) \times \sin \left(\frac{\overline{A}+\overline{a}-m}{2}\right) : \sin A \times \sin a.$$

(32) Theorem 3. Of the circular parts of an oblique spherical triangle, The square of the cosine of half the middle part is to the square of the radius; as the product of the sines of half the sum and half the difference of the remote part and the difference of the adjacent parts, is to the product of the sines of the adjacent parts.

Dem. For since $\sin VB : R^2 :: \sin V, AC - \sin V, (AB - BC) : \sin AB + \sin BC (10)$, it follows that $\cos^2 \frac{1}{2} \text{ supp. } B : R^2 :: \sin \frac{\overline{AC}+\overline{AB}-\overline{BC}}{2} \times \sin \left(\frac{\overline{AC}-\overline{AB}-\overline{BC}}{2}\right) : AB \times \sin BC (2) \text{ and } (1); \text{ therefore (20)}$

$$\cos^2 \frac{1}{2} M : R^2 :: \sin \left(\frac{m+\overline{A}-\overline{a}}{2}\right) \times \sin \left(\frac{m-\overline{A}-\overline{a}}{2}\right) : \sin A \times \sin a.$$

(33) Theorem 4. Of the circular parts of an oblique spherical triangle, The square of the tangent of half the middle part is to the square of the radius; as the product of the sines of half the sum and half the difference of the sum of the adjacent parts and the remote part, is to the product of half the sum and half the difference of the remote part and the difference of the adjacent parts.

That is (by comparing the two preceding theorems)

$$\tan^2 \frac{1}{2} M : R^2 :: \sin \left(\frac{\overline{A}+\overline{a}+m}{2}\right) \times \sin \left(\frac{\overline{A}+\overline{a}-m}{2}\right) : \sin \left(\frac{m+\overline{A}-\overline{a}}{2}\right) \times \sin \left(\frac{m-\overline{A}-\overline{a}}{2}\right)$$

(34) Theorem 5. Of the circular parts of an oblique spherical triangle, The product of the tangents of half the sum and half the difference of the segments of the middle part is equal to the product of the tangents of half the sum and half the difference of the opposite parts.

Dem. For since $\cos BA : \cos BC :: \cos DA : \cos DC (27)$ it follows that $\cos BA + \cos BC : \cos BA - \cos BC :: \cos DA + \cos DC : \cos DA -$

\cos

$\cot DC$; therefore (3) $\cot \left(\frac{BC+BA}{2}\right) : \tan \left(\frac{BC-BA}{2}\right) :: \cot \left(\frac{DC+DA}{2}\right) : \tan \left(\frac{DC-DA}{2}\right)$;
 therefore $\tan \left(\frac{DC+DA}{2}\right) \times \tan \left(\frac{DC-DA}{2}\right) :: \tan \left(\frac{BC+BA}{2}\right) \times \tan \left(\frac{BC-BA}{2}\right)$;
 therefore (20 and 21)

$$\tan \left(\frac{s+r}{2}\right) \times \tan \left(\frac{s-r}{2}\right) = \tan \left(\frac{o+r}{2}\right) \times \tan \left(\frac{o-r}{2}\right)$$

(35) Rem. 4. By any of the theorems 2, 3, or 4, being given the three sides or three angles of a spherical triangle, may be found any of its angles or sides; and, confining the middle part to the supplement of an angle, being given the three sides of a plane triangle, may be found (22)

(36) Rem. 5. By theorem 5, being given the three sides or three angles of a spherical triangle, the segment of any of its sides or angles may be found; and confining the middle part to a side, being given the three sides of a plane triangle, the segments of any of its sides may be found.

(37) Rem. 6. By the first theorem, and any one of the other four, may be solved all the possible cases of spherical and plane triangles. Of these four, the last is the most elegant and the most easily remembered.

(38) Theorem 6. Of the circular parts of an oblique spherical triangle, the tangents of half the sum and half the difference of the segments of the middle part are proportional to the sines of the sum and the difference of the adjacent parts.

Dem. For since $\sin CD : \sin DA :: \cot C : \cot A$ (26), it follows that $\sin CD + \sin DA : \sin CD - \sin DA :: \cot C + \cot A : \cot C - \cot A$; therefore $\tan \left(\frac{CD+DA}{2}\right) : \tan \left(\frac{CD-DA}{2}\right) :: \sin (A+C) : \sin (A-C)$; therefore (20 and 21)

$$\tan \left(\frac{s+r}{2}\right) : \tan \left(\frac{s-r}{2}\right) :: \sin (A+\alpha) : \sin (A-\alpha)$$

(39)

(39) Rem. 7. By this theorem, being given two sides and the included angle, or two angles and the included side of any triangle, the segments of the angle or side may be found.

(40) Theorem 7. Of the circular parts of an oblique spherical triangle, The tangents of half the sum and half the difference of the adjacent parts are proportional to the tangents of half the sum and half the difference of the opposite parts.

Dem. For since $\sin BC : \sin BA :: \sin A : \sin C$ (25), it follows that $\sin BC + \sin BA : \sin BC - \sin BA :: \sin A + \sin C : \sin A - \sin C$, therefore (4) $\tan(\frac{BC+BA}{2}) : \tan(\frac{BC-BA}{2}) :: \tan(\frac{A+C}{2}) : \tan(\frac{A-C}{2})$: therefore (20)

$$\tan(\frac{A+\alpha}{2}) : \tan(\frac{A-\alpha}{2}) :: \tan(\frac{\sigma+\sigma}{2}) : \tan(\frac{\sigma-\sigma}{2}).$$

(41) Rem. 8. By this theorem, being given two sides and the included angle of a plane triangle (22), the other angles may be found.

(42) Theorem 8. Of the circular parts of any spherical triangle, The tangents of half the middle part and half the difference of the opposite parts are proportional to the sines of half the sum and half the difference of the adjacent parts..

Dem. For since $\tan(\frac{s+s}{2}) \times \tan(\frac{s-s}{2}) = \tan(\frac{\sigma+\sigma}{2}) \times \tan(\frac{\sigma-\sigma}{2})$, (34); and $\tan(\frac{s+s}{2}) : \tan(\frac{s-s}{2}) :: \sin(A+\alpha) : \sin(A-\alpha)$, (38); and, $\tan(\frac{\sigma+\sigma}{2}) : \tan(\frac{\sigma-\sigma}{2}) :: \tan(\frac{A+\alpha}{2}) : \tan(\frac{A-\alpha}{2})$, (40) it follows, that $\tan(\frac{s+s}{2}) : \tan(\frac{s-s}{2}) :: \sin(A+\alpha) \times \tan(\frac{A+\alpha}{2}) : \sin(A-\alpha) \times \tan(\frac{A-\alpha}{2})$; therefore (6)

$$\tan(\frac{s+s}{2}) : \tan(\frac{s-s}{2}) :: \sin(A+\alpha) \times \tan(\frac{A+\alpha}{2}) : \sin(A-\alpha) \times \tan(\frac{A-\alpha}{2}).$$

(43)

(43) Theorem 9. Of the circular parts of an oblique spherical triangle, The tangents of half the middle part and half the sum of the opposite parts are proportional to the cosines of half the sum and half the difference of the adjacent parts.

Dem. For since $\tan(\frac{s+s}{2}) \times \tan(\frac{s-s}{2}) = \tan(\frac{o+o}{2}) \times \tan(\frac{o-o}{2})$, (34); and $\tan(\frac{s+s}{2}) : \tan(\frac{s-s}{2}) :: \sin(A+a) : \sin(A-a)$, (38); and $\tan(\frac{o+o}{2}) : \tan(\frac{o-o}{2}) :: \tan(\frac{A-a}{2}) : \tan(\frac{A+a}{2})$ (40); it follows that $\tan(\frac{s+s}{2}) : \tan(\frac{o+o}{2}) :: \sin(A+a) \sin(\frac{A-a}{2}) : \sin(A-a) \tan(\frac{A+a}{2})$: therefore (7)

$$\text{Tang}_x M : \tan(\frac{o+o}{2}) :: \cos(\frac{A+a}{2}) : \cos(\frac{A-a}{2}).$$

(44) Rem. 9. From these two theorems it is evident, that, being given two angles and the included side, or two sides and the included angles of any spherical triangle, the other two sides, or the other two angles may be found; and being given two angles and the included side of any plane triangle, the other two sides may be found by *two* analogies only.

From these propositions are deduced the following

TRIGONOMETRICAL FORMULÆ.

(45) In any spherical triangle ABC, Fig. XIV. we have

$$\sin AB \times \sin BC : R^2 :: \sin \frac{AC + \overline{AB} - \overline{BC}}{2} \times \sin \frac{AC - \overline{AB} - \overline{BC}}{2} : \sin^2 \frac{1}{2} B \quad (32)$$

$$\sin AB \times \sin BC : R^2 :: \sin \frac{\overline{AB} + \overline{BC} + AC}{2} \times \sin \frac{\overline{AB} + \overline{BC} - AC}{2} : \cos^2 \frac{1}{2} B, \quad (31)$$

$$\sin \frac{\overline{AB} + \overline{BC} + AC}{2} \times \sin \frac{\overline{AB} + \overline{BC} - AC}{2} : R^2 :: \sin \frac{AC + \overline{AB} - \overline{BC}}{2} \times \sin \frac{AC - \overline{AB} - \overline{BC}}{2} : \tan^2 \frac{1}{2} B \quad (33)$$

$$\sin A \times \sin C : R^2 :: -\cos \frac{\overline{A} + \overline{C} + B}{2} \times \cos \frac{\overline{A} + \overline{C} - B}{2} : \sin^2 \frac{1}{2} AC$$

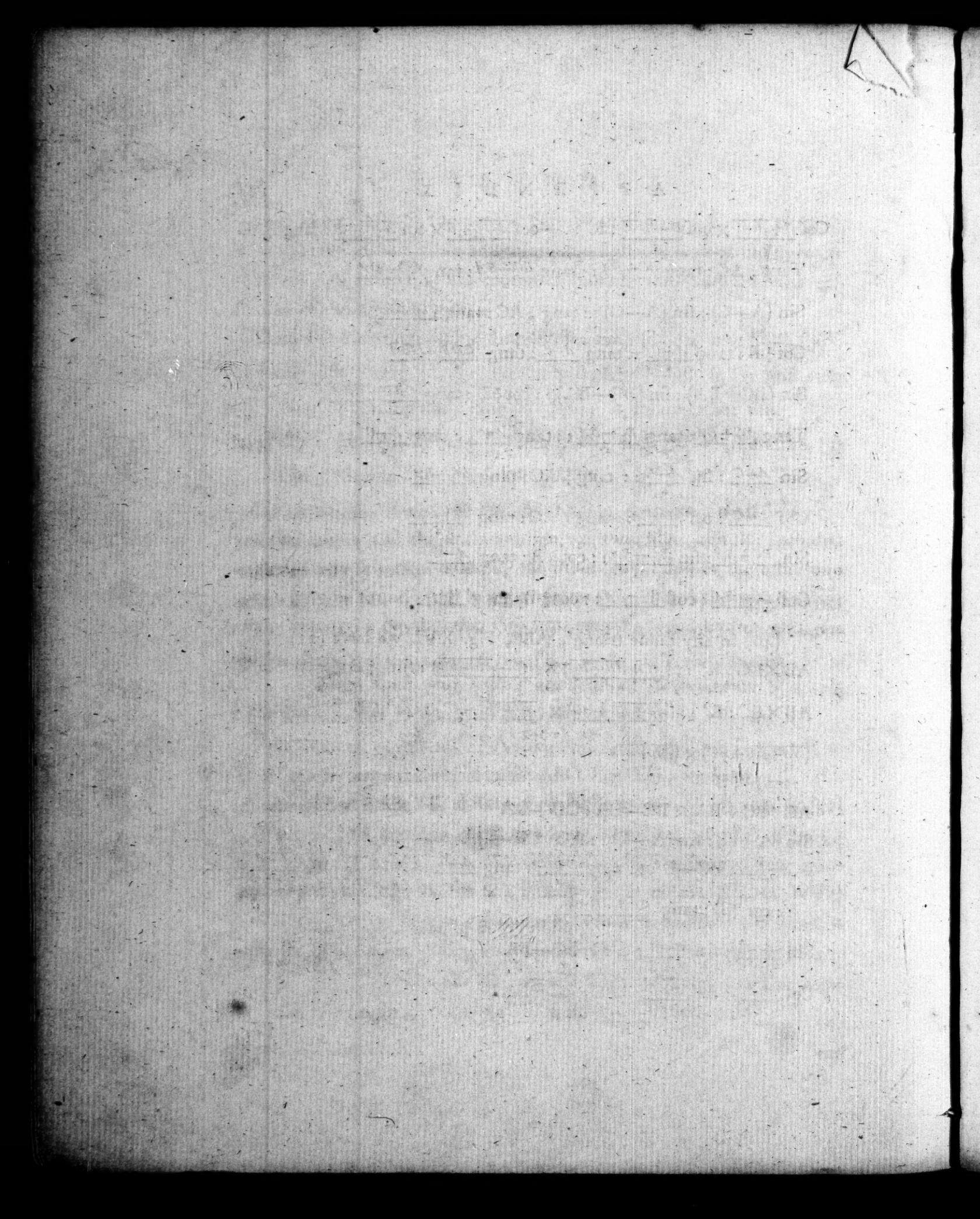
$$\sin A \times \sin C : R^2 :: \cos \frac{\overline{B} + \overline{A} - \overline{C}}{2} \times \cos \frac{\overline{B} - \overline{A} - \overline{C}}{2} : \cos^2 \frac{1}{2} AC$$

cof

$$\begin{aligned}
 & \text{Cof } \frac{B+A-C}{2} \times \text{cof } \frac{B-A-C}{2} : R^2 :: -\text{cof } \frac{A+C+B}{2} \times \text{cof } \frac{A+C-B}{2} : \tan^2 \frac{1}{2} AC \\
 & \text{Tang } \frac{1}{2} AC : \tan \frac{BC+BA}{2} :: \tan \frac{BC-BA}{2} : \tan \frac{CD-DA}{2} \\
 & \text{Sin } (A+C) : \sin (A-C) :: \tan \frac{1}{2} AC : \tan \frac{CD-DA}{2} \\
 & \text{Cot } \frac{1}{2} B : \tan \frac{A+C}{2} :: \tan \frac{A-C}{2} : \tan \frac{CDB-DBA}{2} \\
 & \text{Sin } (BC+BA) : \sin (BC-BA) :: \cot \frac{1}{2} B : \tan \frac{CBD-DBA}{2} \\
 & \text{Tang } \frac{BC+BA}{2} : \tan \frac{BC-BA}{2} :: \tan \frac{A+C}{2} : \tan \frac{A-C}{2} \\
 & \text{Sin } \frac{A+C}{2} : \sin \frac{A-C}{2} :: \tan \frac{1}{2} AC : \tan \frac{BC-BA}{2} \\
 & \text{Cof } \frac{A+C}{2} : \text{cof } \frac{A-C}{2} :: \tan \frac{1}{2} AC : \tan \frac{BC+BA}{2} \\
 & \text{Sin } \frac{BC+BA}{2} : \sin \frac{BC-BA}{2} :: \cot \frac{1}{2} B : \tan \frac{A-C}{2} \\
 & \text{Cof } \frac{BC+BA}{2} : \text{cof } \frac{BC-BA}{2} :: \cot \frac{1}{2} B : \tan \frac{A+C}{2}.
 \end{aligned}$$

(46) In any plane triangle ABC, Fig. XVI. we have (22)

$$\begin{aligned}
 & AB \times BC : R^2 :: (\frac{AC+\overline{AB-BC}}{2}) \times (\frac{AC-\overline{AB-BC}}{2}) : \sin^2 \frac{1}{2} B \\
 & AB \times BC : R^2 :: (\frac{\overline{AB+BC+AC}}{2}) \times (\frac{\overline{AB+BC-AC}}{2}) : \text{cof}^2 \frac{1}{2} B \\
 & (\overline{AB+BC+AC}) \times (\overline{AB+BC-AC}) : R^2 :: (AC+\overline{AB-BC}) \times (AC-\overline{AB-BC}) \\
 & : \tan^2 \frac{1}{2} B \\
 & AC : BC+BA :: BC-BA : CD-DA \\
 & \text{Sin } (A+C) : \sin (A-C) :: AC : CD-DA \\
 & BC+BA : BC-BA :: \tan \frac{A+C}{2} : \tan \frac{A-C}{2} :: \cot \frac{1}{2} B : \tan \frac{A-C}{2} \\
 & :: \cot \frac{1}{2} B : \tan \frac{CDB-DBA}{2} \\
 & \text{Sin } \frac{A+C}{2} : \sin \frac{A-C}{2} :: AC : BC-BA \\
 & \text{Cof } \frac{A+C}{2} : \text{cof } \frac{A-C}{2} :: AC : BC+BA.
 \end{aligned}$$



IV.

THE HYPERBOLA AS CONNECTED WITH THE LOGARITHMS.

1. WHILE a straight line PM (Fig. XVII.) moves parallel to itself along the indefinite straight line CPD with a velocity always proportional to the distance of its extremity P from a fixed point C, let its other extremity M approach to or recede from P, so that PM may describe equal spaces in equal times: The point P will describe a part PP' or Pp' of the straight line CD, while the point M describes a corresponding part MM' or Mm of the curve $m'SM'$.

2. If the motion is supposed to have begun at P, the arca $PM M'P'$ or $PM m'p'$ is the logarithm of the absciss CP' or Cp' .

3. In order that equal spaces may be described in equal times, it is evident that the greater or smaller the absciss CP' or Cp' becomes with regard to CP , the smaller or greater must the ordinate $P'M'$ or $p'm'$ become with regard to PM ; Therefore $CP' : CP :: PM : P'M'$, or $Cp' : CP :: PM : p'm'$; Therefore the product of any absciss by the correspondent ordinate is a constant quantity: Therefore

4. The curve $m'SM'$ is a hyperbola having CD for one of its asymptotes, and Cs , parallel to the ordinates, for the other.

5. From this manner of conceiving the generation of the hyperbola might be deduced the properties of that curve and of the logarithms. That CD and CS, for instance, touch the curve at an infinite distance from C appears from this: When the absciss is infinite, the ordinate must be zero, and when the absciss is zero, the ordinate must be infinite, in order that their product may equal the finite quantity PM \times CP: And that the logarithm of CP is zero appears from this; PM is length without breadth and therefore no space.

6. Let CP = a , PM = μ , PP' = x and PM' = y ; we have (3) $y = \frac{a\mu}{a+x}$, or, developing the fraction $\frac{a}{a+x}$ in the manner first taught by Nicolas Mercator*,

$$Y = \mu \left(1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \&c \right)$$

7. It is evident that the space PMMP' is equal to the sum of all the ordinates $y' + y'' + y''' + \&c.$ on the absciss x . If the absciss is supposed to be divided into an infinite number of infinitely small and equal parts, the abscissæ corresponding to the ordinates $y', y'', y''', \&c.$ may be called 1, 2, 3, &c: therefore (6)

$$y' = \mu \left(1 - \frac{1}{a} + \frac{1}{a^2} - \frac{1}{a^3} + \&c \right)$$

$$y'' = \mu \left(1 - \frac{2}{a} + \frac{2^2}{a^2} - \frac{2^3}{a^3} + \&c \right)$$

$$y''' = \mu \left(1 - \frac{3}{a} + \frac{3^2}{a^2} - \frac{3^3}{a^3} + \&c \right)$$

$$y^* = y = \mu \left(1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \&c \right)$$

therefore

$$y' + y'' + y''' + \&c \dots + y = \mu \left\{ \begin{array}{l} \left. \begin{array}{l} + \frac{1^0}{a^0} + \frac{2^2}{a^0} + \frac{3^0}{a^0} + \&c \dots + \frac{x^0}{a^0} \\ - \frac{1^1}{a^1} - \frac{2^1}{a^1} - \frac{3^1}{a^1} - \&c \dots - \frac{x^1}{a^1} \end{array} \right\} \\ \left. \begin{array}{l} + \frac{1^2}{a^2} + \frac{2^2}{a^2} + \frac{3^2}{a^2} + \&c \dots + \frac{x^2}{a^2} \\ - \frac{1^3}{a^3} - \frac{2^3}{a^3} - \frac{3^3}{a^3} - \&c \dots - \frac{x^3}{a^3} \end{array} \right\} \\ + \&c, \&c. \end{array} \right\}$$

Now,

Now, as was first demonstrated by Wallis*, the sum $1^n + 2^n + 3^n + \&c.$ continued to infinity, that is to x^n in this case, being equal to $\frac{x^n + 1}{n + 1}$; we have

$$PMM'P' = L(a+x) = \mu\left(x - \frac{x^2}{2a} + \frac{x^3}{3a^2} - \&c\right)$$

and, if x is negative,

$$PMm'p' = L(a-x) = -\mu\left(x + \frac{x^2}{2a} + \frac{x^3}{3a^2} + \&c\right)$$

8. The quantity μ depends on the angle $DCs = \Phi$ formed by the asymptotes and the distance $MN = m$ of the point M of the curve from the asymptote CD ; as is evident from its value $\mu = \frac{rm}{\sin \Phi}$, where r denotes the radius of the circle.

* Arith. Infinit.

SECTION.

V.

PROPERTIES OF THE LOGARITHMIC.

1. WHILE two points A and B are in S, (Fig. XVIII.) moving in opposite directions along the indefinite straight line CSD with a velocity always proportional to their distance from a fixed point C, let all the points in SD and all the points in SC move in opposite directions perpendicularly to CSD with any uniform velocity ; and in the instant that A or B passes through any point P' or p' let the point which left P' or p' stop in M' or m' ; A and B will describe the axis, while the points that move perpendicular to it, describe all the ordinates, or the area of the curve $m'SM'$.

2. This curve is called the logarithmic, because its ordinate PM, P' M' &c. are the logarithms of its abscissæ CP, CP', &c.

3. The ordinate C_s , at the finite extremity C of the axis, is an asymptote to the curve : for, as the point that moves from S towards C cannot arrive at C in any finite time, the point that left C will move on for ever.

4. The ordinate PM, a tangent to the curve at whose extremity M meets the point C, is called the modulus of the logarithmic. We shall call PM the logarithmic modulus and CS the numeric modulus.

5. Let the portions $P\pi$ and $P'\pi'$ of the axis be supposed described in equal times; and let the straight lines $M\pi$, $M\pi'$ be drawn perpendicular to the ordinates $\pi\mu$, $\pi'\mu'$: we have $CP : CP' :: P\pi : P'\pi'$ and $\nu\mu = \nu'\mu'$: But, if the equal times are infinitely small, the arcs $M\mu$ and $M'\mu'$ are straight lines and the right angled triangles CPM and $M\mu\mu$, similar; consequently $CP : PM :: M\pi$ or $P\pi : \nu\mu$; therefore $CP' : PM :: P'\pi' : \nu\mu$ or $\nu'\mu'$.

6. *To draw a tangent to any point M' of the Logarithmic.* Upon the ordinate $P'M'$ take $P'L' = PM$; join the points C and L' and draw parallel to CL' the straight line $M'\tau'$ meeting the axis in the point τ' ; $\tau'M'$ touches the curve in the point M' : For since (5) $CP : PM$ or $P'L' : P'\pi'$ or $M'\nu' : \nu\mu'$, the triangles $CP'L'$ and $M'\nu'\mu'$ are similar; therefore $M'\mu'$ is parallel to CL' ; therefore &c: Hence,

7. The ordinates to the asymptote, MQ , MQ' , &c. have for their logarithms its abscissæ CQ , CQ' ; and

8. The subtangents CQ , CQ' , &c. upon the asymptote are all equal to the logarithmic modulus PM .

9. The subtangent TP' upon the axis is to the ordinate PM' as the abscissæ CP' is to the modulus PM ; For the triangles $TP'M'$ and $C'P'M'$ are similar: Hence,

10. The subtangents upon the axis are to each other as the products of the abscissæ and ordinates.

11. The subnormal $P'N'$ upon the axis is to the ordinate as the logarithmic modulus to the abscissæ: For the triangles CPL' and $M'PN'$ are similar: Hence,

12. The subnormals upon the axis are to each other as the quotients of the ordinates and abscissæ.

13. The subtangent is to the subnormal as the square of the absciss to the square of the logarithmic modulus.

14. Let $PM = \mu$, $CP' = z$ and $P'M' = y$: and let $z', z^2, z^3, \dots z^n$ be any number of abscissæ in geometrical progression; $S, S'', S''', \dots S^n$, the correspondent subtangents, and $\sigma', \sigma'', \sigma''', \dots \sigma^n$, the correspondent subnormals upon the axis: we have, (10) and (12)

$$\begin{aligned} n(n+1)(2n+1)y^2 &= 6(S'\sigma' + S''\sigma'' + S'''\sigma''' + \dots + S^n\sigma^n) \\ \frac{z^2}{\mu^2} \left(\frac{1+z''}{1+z} \right) \left(\frac{1-z''}{1-z} \right) &= \frac{S'}{\sigma'} + \frac{S''}{\sigma''} + \frac{S'''}{\sigma'''} + \dots + \frac{S^n}{\sigma^n} \\ y^{2n} &= \frac{S'\sigma'}{1^2} \times \frac{S''\sigma''}{2^2} \times \frac{S'''}{3^2} \times \dots \times \frac{S^n\sigma^n}{n^2} \\ \left(\frac{z^n \times 1}{\mu^n} \right)' &= \frac{S'}{\sigma'} \times \frac{S''}{\sigma''} \times \frac{S'''}{\sigma'''} \times \dots \times \frac{S^n}{\sigma^n} \end{aligned}$$

15. Let the numeric modulus $CS = m$ and $SP' = x$, the denominations of PM and $P'M'$ continuing as before: we have $P'x = x$ and $\mu' = y$. Now $m+x : \mu : : x : y$, (5); therefore $y = \frac{\mu x}{m+x}$; or if $m=1$ $y = \mu x \left(\frac{1}{1+x} \right) = \mu x (1-x+x^2-x^3+\&c)$ and therefore

$$y = \mu \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \&c \right) = \text{Log. } (1+x).$$

16. Let the area of any portion $S'M'P'$ of the curve $= A$: we have $A = zy$; therefore $A = \int zy = zy - \int yz$: but $y = \frac{\mu z}{z}$ (5); or (15); therefore $\int yz = \int \mu z = \mu z$; therefore $A = zy - \mu z + C$: but when $A = 0$, then $z = m$ and $y = 0$; therefore $0 = -\mu m + C$; therefore $C = \mu m$ and $A = z(y-h) + \mu m$, that is

17. The area of any portion of the logarithmic is equal to the rectangle under the absciss and the difference of the ordinate and the logarithmic

garithmic modulus, together with the rectangle under the moduli :
Hence

18. The rectangle CL, under the moduli, is equal to the area SMP contained by the logarithmic modulus PM, the portion of the axis MS, and the arc SM ; or to the area SmC contained by the numeric modulus SC, the asymptote C, and the infinite branch mS of the curve.

F I N I S.



Fig. I.

Four faces of the 1.st Rod

0	1
0	1
0	2
0	3
0	4
0	5
0	6
0	7
0	8
0	9
6	8

Four faces of the 2.nd Rod

0	2
0	2
0	4
0	6
0	8
0	10
0	2
0	4
0	6
0	8
6	4

Four faces of the 3.rd Rod

0	3
0	3
0	6
0	9
0	12
0	3
0	6
0	9
0	12
6	9

Four faces of the 4.th Rod

0	4
0	4
0	8
0	12
0	4
0	12
0	8
0	4
0	12
6	4

Four faces of the 5.th Rod

1	2
1	2
2	4
3	6
4	8
5	10
6	12
7	14
8	16
9	18
8	14
7	12
6	10
5	8
4	6
3	4
2	2

Four faces of the 6.th Rod

1	3
2	6
3	9
4	12
5	15
6	18
7	21
8	24
9	27
8	21
7	15
6	12
5	9
4	6
3	3

Four faces of the 7.th Rod

1	4
2	8
3	12
4	16
5	20
6	24
7	28
8	32
9	36
8	28
7	24
6	20
5	16
4	12
3	8
2	4

Four faces of the 8.th Rod

2	3
4	6
6	9
8	12
10	15
12	18
14	21
16	24
18	27
10	15
8	12
6	9
4	6
2	3

Four faces of the 9.th Rod

2	4
4	8
6	12
8	16
10	20
12	24
14	28
16	32
18	36
10	15
8	12
6	9
4	6
2	4

Four faces of the 10.th Rod

3	4
6	8
9	12
12	16
15	20
18	24
21	28
24	32
27	36
18	24
15	20
12	16
9	12
6	8
3	4

Fig. II.

1	7	8	5	1
2	14	16	10	2
3	21	24	15	3
4	28	32	20	4
5	35	40	25	5
6	42	48	30	6
7	49	56	35	7
8	56	64	40	8
9	63	72	45	9

Fig. III.

1	3	6	4
2	6	12	8
3	9	18	12
4	12	24	16
5	15	30	20
6	18	36	24
7	21	42	28
8	24	48	32
9	27	54	36

Fig. IV.

For the Square	For the Cube
0	1
0	2
0	3
0	4
1	9
2	16
3	25
4	36
5	49
6	64
7	81
8	100
9	121

Fig. V.

b	c	d
c	d	e
d	e	f
e	f	g





Fig. VI.

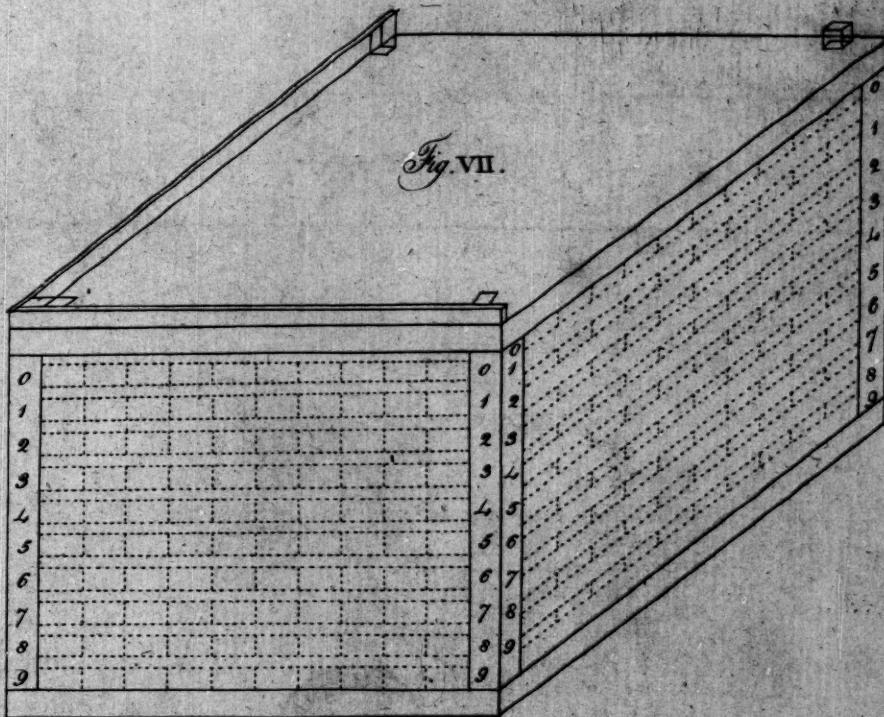


Fig. VII.

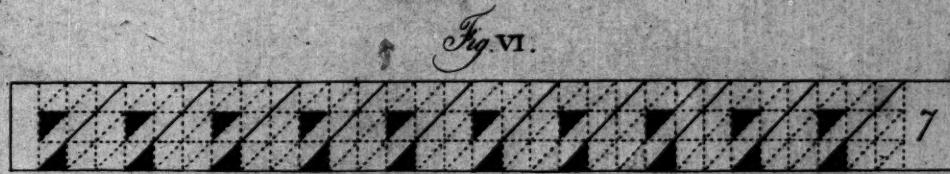


Fig. VI.

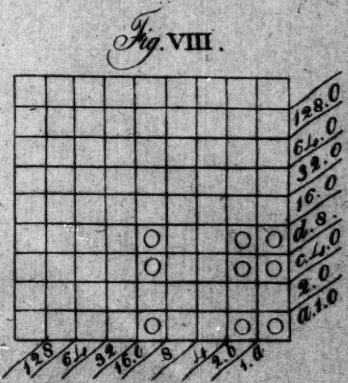


Fig. VIII.

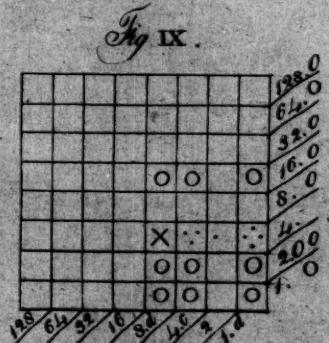


Fig. IX.

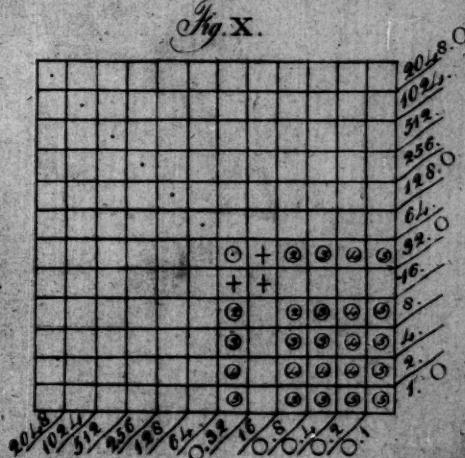
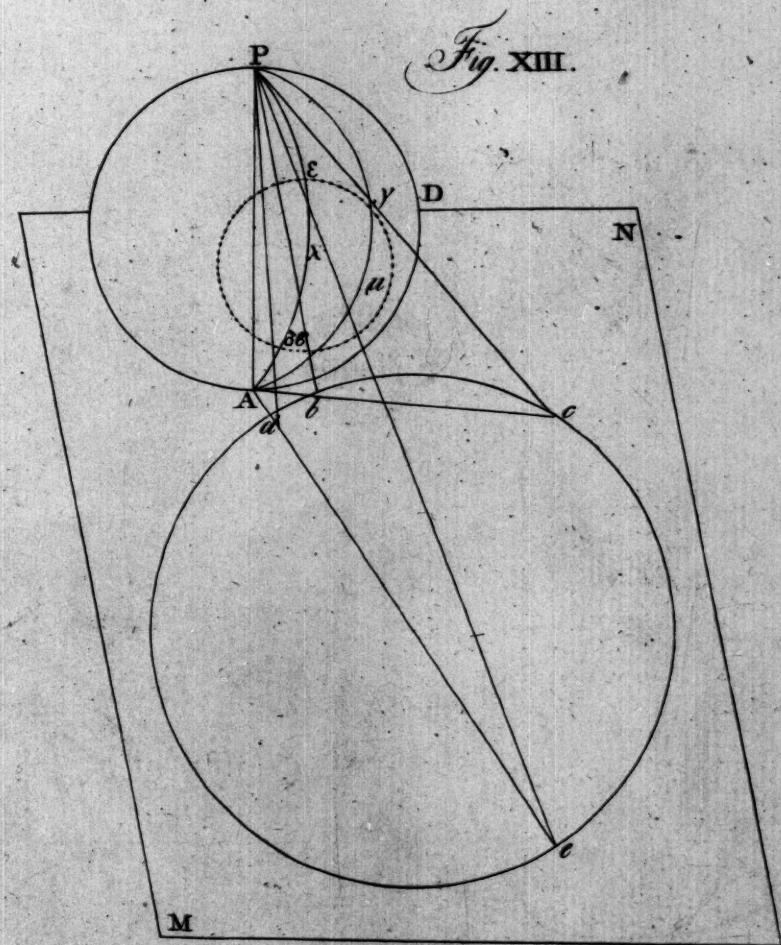
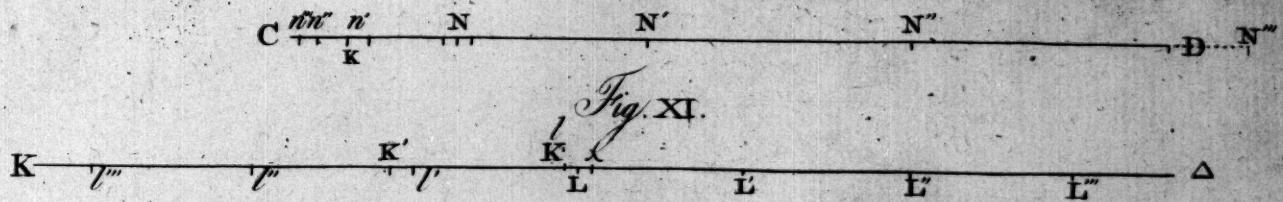


Fig. X.

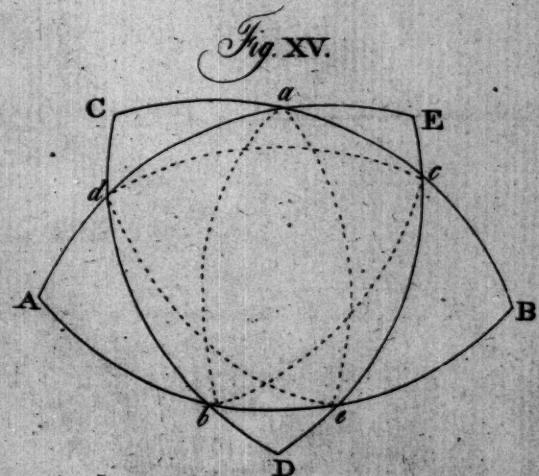
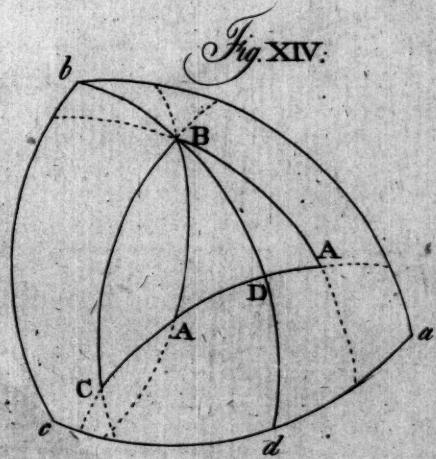
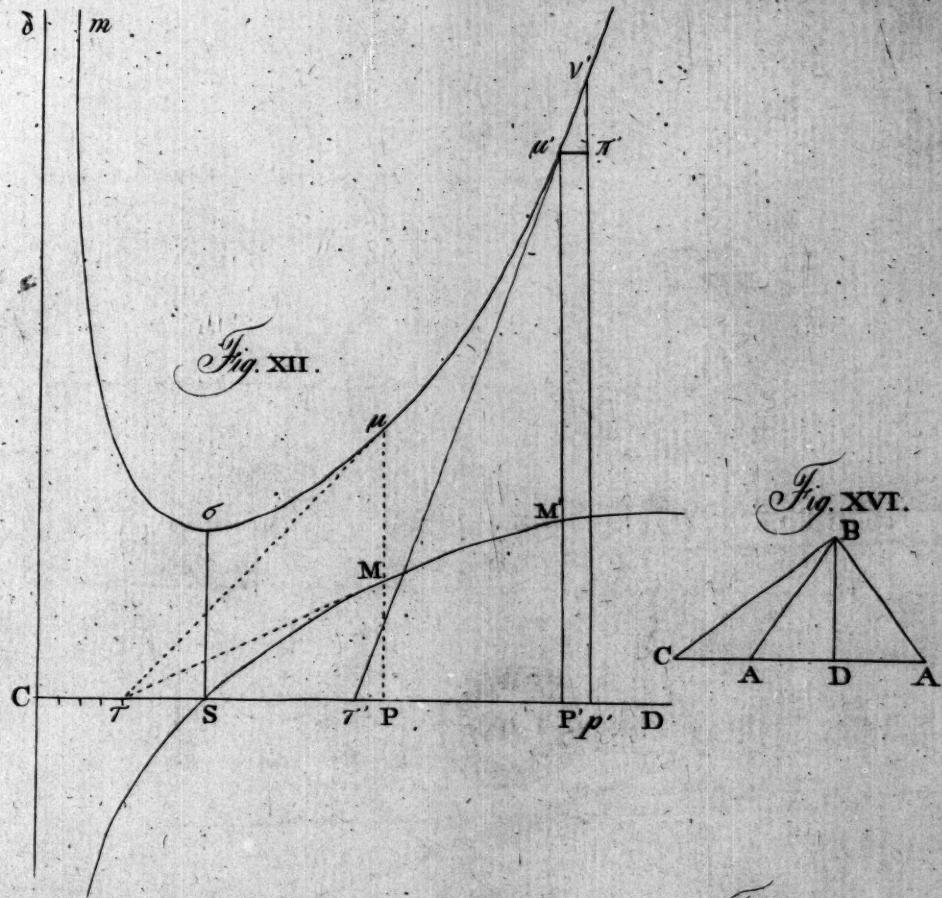


Engraved for the Life of NAPIER

Plate III.









Engraved for the Life of NAPIER

Plate V

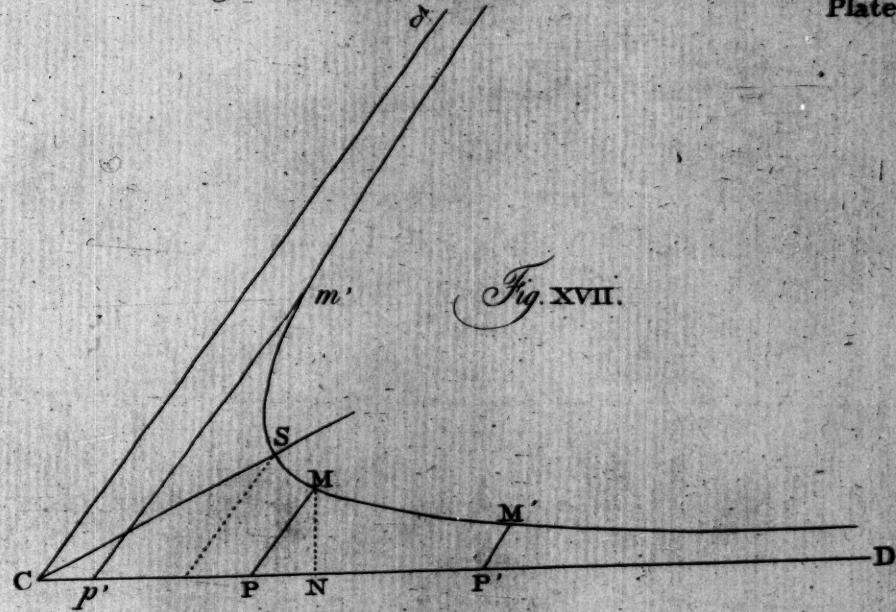


Fig. XVII.

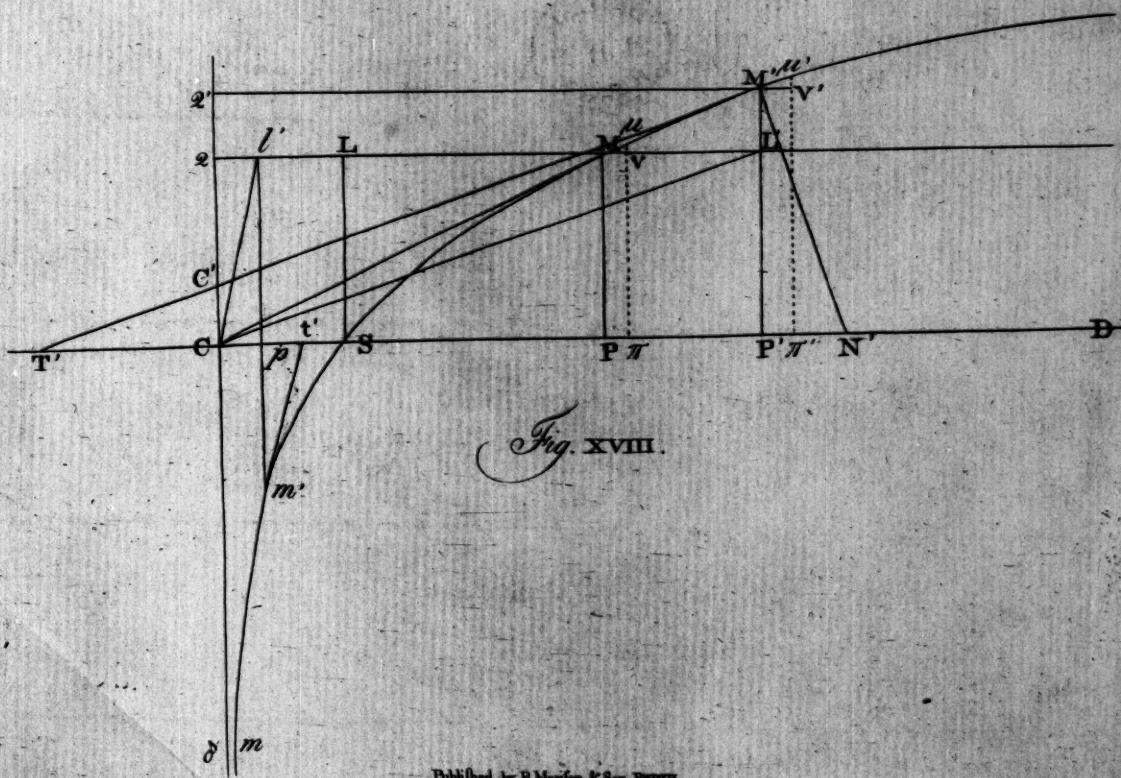


Fig. XVIII.



